

Spatial mixing and approximate counting for Potts model on graphs with bounded average degree

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Abstract

We propose a notion of contraction function for a family of graphs and establish its connection to the strong spatial mixing for spin systems. More specifically, we show that for anti-ferromagnetic Potts model on families of graphs characterized by a specific contraction function, the model exhibits strong spatial mixing, and if further the graphs exhibit certain local sparsity which are very natural and easy to satisfy by typical sparse graphs, then we also have FPTAS for computing the partition function.

This new characterization of strong spatial mixing of multi-spin system does not require maximum degree of the graphs to be bounded, but instead it relates the decay of correlation of the model to a notion of effective average degree measured by the contraction of a function on the family of graphs. It also generalizes other notion of effective average degree which may determine the strong spatial mixing, such as the connective constant [SSY13, SSY15], whose connection to strong spatial mixing is only known for very simple models and is not extendable to general spin systems.

As direct consequences: (1) we obtain FPTAS for the partition function of q -state anti-ferromagnetic Potts model with activity $0 \leq \beta < 1$ on graphs of maximum degree bounded by Δ when $q > 3(1 - \beta)\Delta + 1$, improving the previous best bound $\beta > 3(1 - \beta)\Delta$ [LY13] and asymptotically approaching the inapproximability threshold $q = (1 - \beta)\Delta$ [GvV13]; and (2) we obtain an efficient sampler (in the same sense of fully polynomial-time almost uniform sampler, FPAUS) for the Potts model on Erdős-Rényi random graph $\mathcal{G}(n, d/n)$ with sufficiently large constant d , provided that $q > 3(1 - \beta)d + 4$. In particular when $\beta = 0$, the sampler becomes an FPAUS for proper q -coloring in $\mathcal{G}(n, d/n)$ with $q > 3d + 4$, improving the current best bound $q > 5.5d$ for FPAUS for q -coloring in $\mathcal{G}(n, d/n)$ [Eft14a].

1 Introduction

Spin systems are idealized models for local interactions with statistical behavior. In Computer Science, spin systems are widely used as a model for counting and sampling problems. The *Potts model* is a class of spin systems parameterized by the number of *spin states* $q \geq 2$ and an *activity* $\beta \geq 0$. Given an undirected graph $G = (V, E)$, a *configuration* is a $\sigma \in [q]^V$ that assigns each vertex in the graph one of the q states in $[q]$. Every configuration σ is assigned by the model with weight $w_G(\sigma) := \prod_{e=uv \in E} \beta^{1(\sigma(u)=\sigma(v))}$. A probability distribution over all configurations, called the *Gibbs measure*, can be naturally defined as $\mu(\sigma) = \frac{w_G(\sigma)}{Z}$ where the normalizing factor $Z = Z(G) = \sum_{\sigma \in [q]^V} w_G(\sigma)$ is the *partition function* in statistical physics. When $\beta < 1$ the

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interacting neighbors favor disagreeing spin states over agreeing ones, and the model is said to be *anti-ferromagnetic*.

The partition function gives a general formulation of counting problems on graphs, whose exact computation is #P-hard. For example when $\beta = 0$, the partition function $Z(G)$ gives the number of proper q -colorings of graph G . There is a substantial body of works on the approximate counting of proper q -colorings of graphs in the context of rapid mixing of a generic random walk, called the *Glauber dynamics* [Jer95, BD97, Vig00, DF03, Mol04, HV03, HV06, Hay03, DFHV04].

An exciting accomplishment in recent years is in relating the approximability of partition function for spin systems to the phase transition of the model on the infinite regular trees, also known as Bethe lattices. Here the exact phase transition property we are concerned with is the presence of the *decay of correlation*, also called the *spatial mixing*: assuming arbitrary possible boundary conditions on all vertices at distance ℓ from the root in the regular tree, the error for the marginal distribution at the root measured by the total variation distance goes to 0 as $\ell \rightarrow \infty$. The decay of correlation of the model on the infinite Δ -regular tree undergoes phase transition as the activity parameter β crosses the critical threshold in terms of q and Δ , called the *uniqueness threshold* as it corresponds to the transition threshold for the uniqueness of Gibbs measures on the infinite regular tree. For proper q -colorings, the uniqueness condition for Gibbs measure on the d -regular tree is $q \geq \Delta + 1$ [Jon02], and for anti-ferromagnetic Potts model with $0 < \beta < 1$, the uniqueness threshold on the Δ -regular tree is conjectured to be $q = (1 - \beta)d$ which was proved to be the threshold for the uniqueness of semi-translation invariant Gibbs measures on d -regular tree [GvV13]. For anti-ferromagnetic 2-spin systems (where $q = 2$), it was settled through a series of works [Wei06, SST12, LLY13, Sly10, GvV12, SS12] that the transition of approximability of the partition function for the model on graphs of bounded maximum degree is precisely characterized by the phase transition of the model on regular trees.

For multi-spin systems (where $q \geq 3$), in a seminal work of Gamarnik and Katz [GK12], the decay of correlation was used to give deterministic FPTAS for counting proper q -colorings of graphs and for computing the partition function of general multi-spin systems. This is the first deterministic approximation algorithm for multi-spin systems and also one of the first few deterministic approximation algorithms for #P-hard counting problems. The specific notion of decay of correlation established in [GK12] is a stronger one, namely the *strong spatial mixing*, where the correlation decay is required to hold even conditioning on an arbitrary configuration partially specified on a subset of vertices. Later in [GKM13], the strong spatial mixing was established for proper q -colorings on graphs of maximum degree bounded by Δ when $q \geq \alpha\Delta + 1$ for $\alpha > \alpha^* \approx 1.763..$, which is best bound known for strong spatial mixing for colorings of graphs of bounded maximum degree. When the parameters of the model are in the nonuniqueness regime, there is long-range correlation. In [GvV13], this was used to establish the inapproximability of the partition function for anti-ferromagnetic Potts model on graphs with maximum degree beyond the uniqueness threshold.

In this paper, we are interested in the spatial mixing and FPTAS for spin systems on graphs with unbounded maximum degree. For some special 2-spin systems such as the hardcore model and Ising model with zero field, this was achieved by relating the decay of correlation property to a notion of effective degree of a family of graphs, called *connective constant*. Roughly speaking, the connective constant of a family \mathcal{G} of graphs is bounded by Δ if for all graphs G from \mathcal{G} the number of self-avoiding walks in G of length ℓ starting from any vertex v is bounded by $n^{O(1)}\Delta^\ell$. In [SSY13, SSŠY15], the exact reliance on the maximum degree by the correlation decay and FPTAS for the hardcore model in [Wei06] was replaced by that on the connective constant. However, as pointed out in [SSŠY15], the approach does not extend to general 2-spin systems. For multi-spin systems, the cases for unbounded maximum degree was studied in the literature mostly in

the context of sampling proper q -colorings of Erdős-Rényi graphs $\mathcal{G}(n, d/n)$ with constant average degree d [DFFV06, ES08, MS10, Eft14a, Eft12, Eft14b].

Our goals are to establish strong spatial mixing and to give FPTAS for multi-spin systems on families of general graphs with unbounded maximum degree that are not restricted to $\mathcal{G}(n, d/n)$. It turns out that for multi-spin systems, in order to achieve these goals we need a more robust way than connective constant to measure the effective average degree. We illustrate the necessity of this by a class of bad instances of graphs called *caterpillars*, which were also considered in [Yin14] and [Eft14b]. The caterpillars as in Figure 1 are paths $P = (v_1, v_2, \dots, v_n)$ with each v_i adjoined with k bristles. Consider in particular the proper q -colorings. For the caterpillars with $k \geq q - 2$, the vertices on bristles can always be fixed in a way to force the vertices on the path to alternate between two colors, which means the strong spatial mixing does not hold on any caterpillar with a $k \geq q - 2$. On the other hand, the connective constant of the family of all caterpillars is only 1. This simple example shows how different multi-spin systems can be from the Ising model on graphs of unbounded maximum degree. It also suggests that for multi-spin systems we should take into account both the branching factor (which measures the long-range growth of self-avoiding walks) and the actual degrees of individual vertices (which locally affect the decay of correlation) when studying the decay of correlation on graphs of unbounded maximum degree.

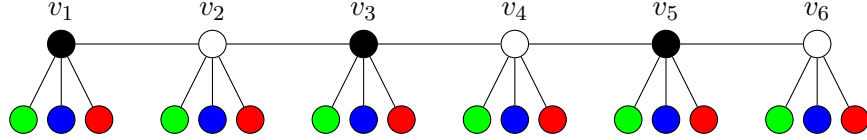


Figure 1: A caterpillar with $n = 6$, $k = 3$ and $q = 5$.

1.1 Contributions

In this paper, we prove strong spatial mixing and give FPTAS for q -state anti-ferromagnetic Potts model and in particular the q -colorings, for families of sparse graphs with unbounded maximum degree. We achieve this by relating strong spatial mixing on a family of graphs to the function which is *contracting* on the graphs from the family, a notion that generalizes the connective constant [SSY13, SSŠY15] and properly measures the effective average degree that affects the decay of correlation in general spin systems.

As connective constant, it is convenient to talk about contraction function on infinite graphs. Given a vertex v in a locally finite infinite graph $G(V, E)$, let $\text{SAW}(v, \ell)$ denote the set of self-avoiding walks in G of length ℓ starting at v . The function $\delta : \mathbb{N} \rightarrow \mathbb{R}^+$ is a *contraction function* for graph G if

$$\sup_{v \in V} \limsup_{\ell \rightarrow \infty} \mathcal{E}_\delta(v, \ell) = 0 \quad \text{where} \quad \mathcal{E}_\delta(v, \ell) := \sum_{\substack{(v, v_i, \dots, v_\ell) \\ \in \text{SAW}(v, \ell)}} \prod_{i=1}^{\ell} \delta(\deg(v_i)),$$

This definition can be naturally extended to a family \mathcal{G} of finite graphs, in such a way that $\delta(\cdot)$ is contracting for \mathcal{G} if $\mathcal{E}_\delta(v, \ell)$ is of exponential decay in ℓ for all $G \in \mathcal{G}$ and any vertex v in G (see Section 2).

We can use the contraction function $\delta(\cdot)$ to describe various families \mathcal{G} of graphs. For example, the families \mathcal{G} of graphs with maximum degree bounded by Δ can be described precisely by the

contraction function such that $\delta(d) = \frac{1}{\Delta}$ if $d \leq \Delta$ and $\delta(d) = \infty$ if otherwise. The contraction function also gives a more robust way than the connective constant to capture average degrees.

Proposition 1. *The families of graphs \mathcal{G} with connective constant bounded strictly by Δ are precisely the families \mathcal{G} for which the constant function $\delta(d) = \frac{1}{\Delta}$ is a contraction function.*

The FPTAS in [SSŠY15] for the hardcore model with activity λ on the families of graphs with bounded connective constant is actually an FPTAS for any family \mathcal{G} of graphs for which a critically defined $\delta(\cdot)$ is a contraction function.¹ It was proved in [SSŠY15] that $\delta(d) \leq \frac{1}{\Delta_c}$ for all d with the equality holds precisely at the critical threshold $d = \Delta_c$, therefore with the notion of contraction function we observe that the FPTAS in [SSŠY15] works on strictly broader families of graphs than what was guaranteed in [SSŠY15] with the connective constant.

Our first result relates the decay of correlation (in the sense of strong spatial mixing) of the anti-ferromagnetic Potts model on a family \mathcal{G} of general graphs to its contraction function.

Theorem 2 (Main theorem: strong spatial mixing). *Let $q \geq 3$ be an integer and $0 \leq \beta < 1$. Let \mathcal{G} be a family of finite graphs that satisfy the followings:*

- *the following $\delta(\cdot)$ is a contraction function for \mathcal{G} :*

$$\delta(d) = \begin{cases} \frac{2(1-\beta)}{q-1-(1-\beta)d} & \text{if } d \leq \frac{q-1}{1-\beta} - 2, \\ 1 & \text{otherwise;} \end{cases} \quad (1)$$

- *(proper q -coloring) if $\beta = 0$, the family \mathcal{G} also needs to be q -colorable.*

Then the q -state Potts model with activity β exhibits strong spatial mixing on all graphs in \mathcal{G} .

Remark. The contraction function describes a notion of average degree and the theorem holds for graphs with unbounded maximum degree. For example, for Erdős-Rényi random graph $\mathcal{G}(n, d/n)$, which with high probability has constant average degree $(1 - o(1))d$ and unbounded maximum degree $\Theta(\log n / \log \log n)$, assuming $q > 3(1 - \beta)d + O(1)$ with high probability the above $\delta(\cdot)$ is a contraction function for $\mathcal{G}(n, d/n)$. Note that $q = (1 - \beta_c)\Delta$ is the uniqueness/nonuniqueness threshold for semi-translation invariant Gibbs measures of Potts model on the infinite Δ -regular tree \mathbb{T}_Δ , which is also conjectured to be the uniqueness/nonuniqueness threshold [GvV13].

In order for the SSM to imply an FPTAS for computing the partition function, we need the graphs to be sparse in a slightly more restrictive manner than what guaranteed by the above contraction function. The conditions of (1) classify the vertices in a graph into low-degree vertices (if $\deg(v) < \frac{q-1}{1-\beta} - 2$) and high-degree vertices (if otherwise). A graph $G = (V, E)$ is said to be *locally sparse* if for every path P in G of length ℓ , the total size of clusters of high-degree vertices growing from path P is bounded by $O(\ell + \log |V|)$ (see Section 2 for a formal definition). Intuitively, a graph is locally sparse if all clusters of high-degree vertices are small (of size $O(\log |V|)$) and are relatively far away from each other.

Theorem 3 (Main theorem: approximate counting). *Let $q \geq 3$ be an integer and $0 \leq \beta < 1$. Let \mathcal{G} be a family of locally sparse graphs satisfying the conditions of Theorem 2. Then there is an FPTAS for the partition function of the q -state Potts model with activity β for all graphs in \mathcal{G}*

¹This $\delta(\cdot)$ is defined formally as follows: $\delta(d) = d^{\frac{1}{\rho}-1} \left(\frac{dx-1}{x+1} \right)^{\frac{1}{\rho}}$, where x is the unique positive solution to $dx = \lambda(1+x)^{-d} + 1$ and $\rho = 2 - (\Delta_c - 1) \ln \left(1 + \frac{1}{\Delta_c - 1} \right)$ where $\Delta_c = \Delta_c(\lambda)$ is the critical (real) degree satisfying $\lambda = \frac{\Delta_c^{\Delta_c}}{(\Delta_c - 1)^{\Delta_c + 1}}$.

Remark. This is the first FPTAS for the general Potts model on families of graphs of unbounded maximum degree. And even for graphs with bounded maximum degree Δ Theorem 3 holds with a much better bound $q > 3(1-\beta)\Delta+1$, which greatly improves the best previous bound $3(1-\beta)\Delta < \beta$ known for anti-ferromagnetic Potts model on graphs with bounded maximum degree [LY13]. Recall that $q = (1 - \beta_c)\Delta$ is the semi-translation invariant uniqueness threshold on the infinite d -regular tree \mathbb{T}_Δ , and the problem is hard to approximate for all even q satisfying $q < (1 - \beta)\Delta$ [GvV13].

To evaluate how tight Theorem 2 and 3 could be, we instead consider an idealized goal: to find a $\delta(\cdot)$ (depending only on the model) such that the strong spatial mixing rate with respect to any v in a graph G is always bounded by $\mathcal{E}_\delta(v, \ell)$. With this stronger requirement, for q -coloring, the following $\delta(\cdot)$ is the best we can hope for:

$$\delta(d) = \begin{cases} \frac{1}{q-d-1} & \text{if } d \leq q-2 \\ 1 & \text{otherwise} \end{cases}$$

because it is achieved by the SSM rate on caterpillars as illustrated in Figure 1 when the colors of leaves are fixed properly. This certainly gives a lower bound to any contraction function for the above idealized goal. Note that the $\delta(\cdot)$ function in (1) in the case of q -coloring (when $\beta = 0$) is precisely twice this lower bound function. And this factor 2 is due to an intrinsic obstacle in the current approaches for correlation-decay based algorithms for multi-spin systems.

As an application of Theorem 3 we consider the Erdős-Rényi random graph $\mathcal{G}(n, d/n)$ with constant average degree d . It is well known that the partition function of the Potts model on this random graph is highly concentrated to its expectation which is easy to calculate [COV13], so for this model sampling is more interesting than counting. And because the FPTAS in Theorem 3 actually works for a broader self-reducible family of instances (for example, the list-colorings instead of just q -colorings), we also have efficient sampling algorithms due to the standard Jerrum-Valiant-Vazirani reduction [JVV86]. So we prove the following result.

Theorem 4. *Let d be sufficiently large and $G \sim \mathcal{G}(n, d/n)$. Let $0 \leq \beta < 1$ and $q > 3(1 - \beta)d + 4$. There exists an algorithm \mathcal{S} such that for any $\epsilon > 0$ with high probability \mathcal{S} returns a random configuration in $[q]^{V(G)}$ from a distribution that is within total variation distance ϵ from the Gibbs distribution μ_G for the q -state Potts model with activity β . And the running time of \mathcal{S} is in polynomial in n and $\log \frac{1}{\epsilon}$. When $\beta = 0$, i.e. for $q > 3d + 4$, with high probability \mathcal{S} is an FPAUS (fully polynomial-time almost uniform sampler) for proper q -colorings of G .*

This is the first result for sampling in general Potts model on $\mathcal{G}(n, d/n)$ and also improves the state of the arts for the special case of sampling proper q -colorings of $\mathcal{G}(n, d/n)$. For this special case of the problem, FPAUSes have been obtained mostly by the rapidly mixing of certain block Glauber dynamics [DFFV06, ES08, MS10, Eft14a]. The best bound $q > 5.5d$ was achieved in [Eft14a], whereas our bound for proper q -coloring is $q > 3d + 4$. Further, the results of [DFFV06, ES08, MS10, Eft14a] are more restricted to $\mathcal{G}(n, d/n)$. In contrast, our algorithm is more generic and works for all families of locally sparse graphs with a bounded growth rate of self-avoiding walks measured properly by the contraction function.

In [Eft12, Eft14b], sampling algorithms with a much weaker control of total variation errors than FPAUS were considered, and with this weaker sampler a better bound $q \geq (1 + \epsilon)d$ was achieved, which almost approaches the uniqueness threshold.

1.2 Techniques

The approximation algorithms for multi-spin systems (e.g. graph coloring) have been studied in the literature in the context of rapid mixing of Glauber dynamics. In a seminal work of Gamarnik

and Katz [GK12], the correlation-decay based deterministic algorithms for multi-spin systems are introduced. Our analysis of decay of correlation utilizes the approach of [GK12] in an essential manner.

Consider for example the proper q -colorings of graph G . If there is a vertex v in G with degree much higher than q , then the Glauber dynamics will have torpid mixing around v since the color of v will be frozen at most of the time; and the bound on the decay of correlation also breaks at this high-degree vertex because locally it may have absolute correlation with the neighbors. Further, for graph coloring with maximum degree unbounded, even the feasibility becomes an issue. These bad situations were dealt with in [DFFV06, MS10, Eft14a] by using block dynamics, where the block contains the high-degree vertices as its core which is separated from the boundary by a buffer of small-degree vertices. As the bound on q getting tighter, the constructions of such blocks have to be highly delicate to meet this requirement.

The novelty in our approach is that we give a correlation-decay based deterministic algorithm that works in terms of blocks. A key observation of us is that despite the correlation can be absolute between a pair of high-degree vertices within the same block, its contribution to the decay rate along a self-avoiding walk is at most a factor 1 (hence the $\delta(d) = 1$ branch for large d in (1)), while the low-degree vertices at the boundaries of blocks contribute to the decay of correlation as in the bounded degree case. While this observation is made to the original Gibbs measure, algorithmically it can be witnessed by applying the recursion in terms of marginal distributions on blocks. In contrast to the block construction in [DFFV06, ES08, MS10, Eft14a], the blocks in our algorithm are extremely simple and generic: they are just clusters of high-degree vertices. This simple construction of blocks makes our algorithm more generic and works on general graphs.

In a previous work [Yin14], this idea of block version of decay of correlation was used to establish a “spatial mixing only” result for random graphs, augmented from the result of Gamarnik *et al.* [GKM13] for graphs of bounded maximum degree. The current work emphasizes the strong spatial mixings that have algorithmic implications. Here the property of being locally sparse is used to bound the running time.

2 Preliminaries

Let $G = (V, E)$ be an undirected graph. For any subset $S \subseteq V$ of vertex, let $G[S]$ denote the subgraph of G induced by S , and let $\partial B = \{u \in V \setminus B \mid \exists w \in B, (u, w) \in E\}$ denote the vertex boundary of B . Given a vertex v in G , let $\text{dist}_G(v, S)$ denote the minimum distance from v to any vertex $u \in S$ in G .

Potts model and spatial mixing. The Potts model is parameterized by an integer $q \geq 2$ and a real $\beta \geq 0$ called the *activity* parameter. Each element of $[q]$ is called a *color* or a *state*. Let $G = (V, E)$ be a graph. A *configuration* $\sigma \in [q]^\Lambda$ on a subset $\Lambda \subseteq V$ of vertices assigns each vertex v in Λ one of the q colors in $[q]$. In the Potts model on graph G , each configuration $\sigma \in [q]^V$ is assigned a weight

$$w_G(\sigma) = \beta^{\#\text{mon}(\sigma)},$$

where $\#\text{mon}(\sigma) = |\{(u, v) \in E \mid \sigma(u) = \sigma(v)\}|$ gives the number of monochromatic (undirected) edges in the configuration σ .

In order to study strong spatial mixing, we consider the instances of Potts model with boundary conditions. An instance of Potts model is a tuple $\Omega = (G, \Lambda, \sigma)$ where $G = (V, E)$ is an undirected graph, $\Lambda \subseteq V$ is a subset of vertices in G and $\sigma \in [q]^\Lambda$ is a configuration on Λ . Given such an instance $\Omega = (G, \Lambda, \sigma)$, the weight function w_Ω assigns each configuration $\pi \in [q]^V$ the weight

$w_\Omega(\pi) = w_G(\pi)$ if π agrees with σ over all vertices in Λ , and $w_\Omega(\pi) = 0$ if otherwise. An instance Ω is *feasible* if there exists a configuration on V with positive weight. This gives rise to a nature probability distribution over all configurations on V for a feasible Potts instance:

$$\mathbf{Pr}_\Omega [c(V) = \pi] = \frac{w_\Omega(\pi)}{Z(\Omega)},$$

where $Z(\Omega) = \sum_{\sigma \in [q]^V} w_\Omega(\sigma)$ is call the *partition function*. This probability distribution is called the *Gibbs measure*. For a vertex $v \in V$ and any color $x \in [q]$, we use $\mathbf{Pr}_\Omega [c(v) = x]$ to denote the marginal probability that v is assigned color x by a configuration sampled from the Gibbs measure. Similarly, for a set $S \subseteq V$ and $\pi \in [q]^S$, we use $\mathbf{Pr}_\Omega [c(S) = \pi]$ to denote the marginal probability that S is assigned configuration π by a configuration sampled from the Gibbs measure. Next we define the notion of strong spatial mixing.

Definition 5 (strong spatial mixing). Let \mathcal{G} be a family of graphs. We say that the q -state Potts model with activity β exhibits *strong spatial mixing* on all graphs in \mathcal{G} if there exist positive constants $C > 0, \gamma < 1$ such that for every $G = (V, E) \in \mathcal{G}$, every $v \in V$, $\Lambda \subseteq V$, $x \in [q]$, and any configurations $\sigma, \tau \in [q]^\Lambda$ that $\Omega_1 = (G, \Lambda, \sigma)$ and $\Omega_2 = (G, \Lambda, \tau)$ are both feasible, it holds that

$$|\mathbf{Pr}_{\Omega_1} [c(v) = x] - \mathbf{Pr}_{\Omega_2} [c(v) = x]| \leq n^C \gamma^{\text{dist}_G(v, \Delta)},$$

where $n = |V|$ and $\Delta \subseteq \Lambda$ is the set of vertices on which σ and τ differ.

If we change $\text{dist}_G(v, \Delta)$ to $\text{dist}_G(v, \Lambda)$ in the definition it becomes the definition of *weak spatial mixing*.

Permissive block and locally sparse. Fix any $q \geq 2$ and $0 \leq \beta < 1$. Let $\Omega = (G, \Lambda, \sigma)$ be an instance of q -state Potts model with activity β and v a vertex in G . We call v a *low-degree* vertex if $\deg_G(v) < \frac{q-1}{1-\beta} - 2$, and otherwise we call it a *high-degree* vertex.

Definition 6 (permissive block). Let $\Omega = (G, \Lambda, \sigma)$ be a Potts instance where $G = (V, E)$. A vertex set $B \subseteq V \setminus \Lambda$ is a *permissive block* in Ω if every boundary vertex $u \in \partial B \setminus \Lambda$ is a low-degree vertex. For any subset of vertices $S \subseteq V \setminus \Lambda$, we denote $B(S) = B_\Omega(S)$ the *minimal permissive block* containing S . And we write $B(v) = B(S)$ if $S = \{v\}$ is a singleton.

Definition 7. A family \mathcal{G} of finite graphs is *locally sparse* if there exists a constant $C > 0$ such that for every $G = (V, E)$ in the family and every path P in G of length ℓ we have $|B(P)| \leq C(\ell + \log |V|)$.

Feasibility and local feasibility. Let $\Omega = (G, \Lambda, \sigma)$ where $G = (V, E)$ be an instance of Potts model, $v \in V \setminus \Lambda$ be a vertex. For a subset of vertices $S \subseteq V \setminus \Lambda$, a configuration $\pi \in [q]^S$ is (globally) *feasible* if there exists a configuration on V with positive weight and agrees with π on S . A configuration $\pi \in [q]^S$ is *locally feasible*, if $w_{G[\Lambda \cup S]}(\sigma \cup \pi) > 0$, where $\sigma \cup \pi$ is the configuration over $\Lambda \cup S$ that agrees with both σ and π .

The discussion of feasibility and local feasibility is meaningful only when $\beta = 0$. In this case, the local feasibility of a configuration on a permissive block implies the (global) feasibility.

Proposition 8. Let $\Omega = (G, \Lambda, \sigma)$ where $G = (V, E)$ be an feasible instance of Potts model with $\beta = 0$, $v \in V \setminus \Lambda$ be a vertex and $\pi \in [q]^{B(v)}$ be a locally feasible configuration. Then π is also feasible.

Proof. Denote $B = B(v)$. Fix a configuration $\eta \in [q]^V$ such that $w_\Omega(\eta) > 0$, this is possible since Ω is feasible. We denote by η' the restriction of η to $V \setminus ((B \cup \partial B) \setminus \Lambda)$, i.e., the set of vertices that are either in Λ , or not in $B \cup \partial B$.

Consider the configuration $\eta = \pi \cup \eta' \in [q]^{V \setminus (\partial B \setminus \Lambda)}$, it can be extended to a configuration $\rho \in [q]^V$ with $w_\Omega(\rho) > 0$ in a greedy fashion, since every vertex in $\partial B \setminus \Lambda$ is of low-degree. Thus ρ witness that π is feasible. \square

With this proposition, we do not distinguish between local feasibility and feasibility of configurations on permissive blocks. For a permissive block B , we use $\mathcal{F}(B)$ to denote the set of feasible configuration. Note that when $\beta > 0$, the set $\mathcal{F}(B)$ is simply $[q]^B$.

Self-avoiding walk tree. Given a graph $G = (V, E)$ and a vertex $v \in V$, a rooted tree T can be naturally constructed from all self-avoiding walks starting from v in G as follows: Each vertex in T corresponds to a self-avoiding walk (simple path in G) $P = (v, v_1, v_2, \dots, v_k)$ starting from v , whose children correspond to all self-avoiding walks $(v, v_1, v_2, \dots, v_k, v_{k+1})$ in G extending P , and the root of T corresponds to the trivial walk (v) . The resulting tree, denoted by $T_{\text{SAW}}(G, v)$, is called the *self-avoiding walk* (SAW) tree constructed from vertex v in graph G .

From this construction, every vertex in $T_{\text{SAW}}(G, v)$ can be naturally identified with the vertex in V (many-to-one) at which the corresponding self-avoiding walk ends.

Connective constant and contraction function. Given a vertex v in a locally finite graph $G(V, E)$, let $\text{SAW}(v, \ell)$ denote the set of self-avoiding walks in G of length ℓ starting at v . The following notion of connective constant of families of finite graphs is introduced in [SSY13].

Definition 9 (connective constant [SSY13, SSŠY15]). Let \mathcal{G} be a family of finite graphs. The *connective constant* of \mathcal{G} is bounded by Δ if there exists a positive constant $C > 0$ such that for any graph $G = (V, E)$ in \mathcal{G} and any vertex v in G , we have $|\text{SAW}(v, \ell)| \leq n^C \Delta^\ell$ where $n = |V|$ for all $\ell \geq 1$.

Let $\delta : \mathbb{N} \rightarrow \mathbb{R}^+$ be a function. Given a vertex v in a locally finite graph $G(V, E)$, let

$$\mathcal{E}_\delta(v, \ell) := \sum_{\substack{(v, v_1, \dots, v_\ell) \\ \in \text{SAW}(v, \ell)}} \prod_{i=1}^{\ell} \delta(\deg(v_i)).$$

Definition 10 (contraction function). Let \mathcal{G} be a family of finite graphs. The $\delta : \mathbb{N} \rightarrow \mathbb{R}^+$ is a *contraction function* for \mathcal{G} if there exist positive constants $C > 0, \gamma < 1$ such that for any graph $G = (V, E)$ in \mathcal{G} and any vertex v in G , we have $\mathcal{E}_\delta(v, \ell) < n^C \gamma^\ell$ where $n = |V|$ for all $\ell \geq 1$.

It is easy to see that graph families \mathcal{G} with constant contraction function $\delta(d) = \frac{1}{\Delta}$ are precisely the families \mathcal{G} of connective constant bounded strictly by Δ .

3 Recursion

In this section, we introduce recursions to compute the marginal probability on a vertex and on a permissive block in Potts model respectively.

Let $\Omega = (G, \Lambda, \sigma)$ where $G = (V, E)$ be an instance of Potts model and $v \in V \setminus \Lambda$ be a vertex. Let $B = B(v)$ be the minimal permissive block containing v . Let $\delta B = \{u_i v_i \mid i \in [m]\}$ be an enumeration of boundary edges of B where $v_i \notin B$ for every $i \in [m]$. In this notation, more than

one u_i or v_i may refer to the same vertex. We denote $E(B) := \{uv \in E \mid u, v \in B\}$ the edges in B . We use \bar{B} to denote the inner boundary of B , i.e., $\bar{B} = \{u \in B \mid uv \in E \text{ and } v \notin B\}$.

Recall that we use $\mathcal{F}(B)$ to denote the set of feasible configurations on a permissive block B , it is easy to see that, for every $x \in [q]$,

$$\Pr_{\Omega} [c(v) = x] = \sum_{\substack{\pi \in \mathcal{F}(B) \\ \pi(v)=x}} \Pr_{\Omega} [c(B) = \pi].$$

This identity relates the marginal probability on a vertex to marginal probabilities on a block. We now define notations for some sub-instances and give a block-to-vertices identity.

Let $\pi \in \mathcal{F}(B)$ be a configuration on a permissive block B . For every $i \in [m]$, denote $\pi_i = \pi(u_i)$. Let $G_B = (V_B, E_B)$ denote the graph obtained from G by removing $B \setminus \bar{B}$ and edges in $E(B)$, i.e., $V' = (V \setminus B) \cup \bar{B}$, $E' = E \setminus E(B)$. Let $\Omega_B = (G_B, \Lambda, \sigma)$. For every $i = 1, 2, \dots, m+1$, define $\Omega_i^{\pi} = (G_i^{\pi}, \Lambda_i^{\pi}, \sigma_i^{\pi})$ as the instance obtained from Ω_B by fixing u_j to color π_j for every $j \in [i-1]$ and by removing edges $u_j v_j$ for every $j = i, i+1, \dots, m$.

Lemma 11. *Assuming above notations, it holds that*

$$\Pr_{\Omega} [c(B) = \pi] = \frac{w_{G[B]}(\pi) \cdot \prod_{i=1}^m (1 - (1 - \beta) \Pr_{\Omega_i^{\pi}} [c(v_i) = \pi_i])}{\sum_{\rho \in \mathcal{F}(B)} w_{G[B]}(\rho) \cdot \prod_{i=1}^m (1 - (1 - \beta) \Pr_{\Omega_i^{\rho}} [c(v_i) = \rho_i])}. \quad (2)$$

Proof.

$$\begin{aligned} \Pr_{\Omega} [c(B) = \pi] &= \frac{w_{G[B]}(\pi) \cdot Z(\Omega_{m+1}^{\pi})}{\sum_{\rho \in \mathcal{F}(B)} w_{G[B]}(\rho) \cdot Z(\Omega_{m+1}^{\rho})} \\ &= \frac{w_{G[B]}(\pi) \cdot \frac{Z(\Omega_{m+1}^{\pi})}{Z(\Omega_1^{\pi})}}{\sum_{\rho \in \mathcal{F}(B)} w_{G[B]}(\rho) \cdot \frac{Z(\Omega_{m+1}^{\rho})}{Z(\Omega_1^{\rho})}} \\ &= \frac{w_{G[B]}(\pi) \cdot \prod_{i=1}^m \frac{Z(\Omega_{i+1}^{\pi})}{Z(\Omega_i^{\pi})}}{\sum_{\rho \in \mathcal{F}(B)} w_{G[B]}(\rho) \cdot \prod_{i=1}^m \frac{Z(\Omega_{i+1}^{\rho})}{Z(\Omega_i^{\rho})}}. \end{aligned}$$

Since for every $\rho \in \mathcal{F}(B)$ and $i \in [d]$,

$$Z(\Omega_{i+1}^{\rho}) = \sum_{y \in [q]} Z(\Omega_i^{\rho} \mid c(v_i) = y) \cdot \beta^{\mathbf{1}(y=\rho(u_i))}$$

where $Z(\Omega_i^{\rho} \mid c(v_i) = y)$ stands for the sum of the weights of all feasible configurations σ on Ω_i^{ρ} satisfying $\sigma(v_i) = y$ and $\mathbf{1}(\cdot)$ is the indicator function. With this identity, we can further write

$$\begin{aligned} \Pr_{\Omega} [c(B) = \pi] &= \frac{w_{G[B]}(\pi) \cdot \prod_{i=1}^m \frac{\sum_{y \in [q]} Z(\Omega_i^{\pi} \mid c(v_i)=y) \cdot \beta^{\mathbf{1}(y=\pi(u_i))}}{Z(\Omega_i^{\pi})}}{\sum_{\rho \in \mathcal{F}(B)} w_{G[B]}(\rho) \cdot \prod_{i=1}^m \frac{\sum_{y \in [q]} Z(\Omega_i^{\rho} \mid c(v_i)=y) \cdot \beta^{\mathbf{1}(y=\rho(u_i))}}{Z(\Omega_i^{\rho})}} \\ &= \frac{w_{G[B]}(\pi) \cdot \prod_{i=1}^m (1 - (1 - \beta) \Pr_{\Omega_i^{\pi}} [c(v_i) = \pi_i])}{\sum_{\rho \in \mathcal{F}(B)} w_{G[B]}(\rho) \cdot \prod_{i=1}^m (1 - (1 - \beta) \Pr_{\Omega_i^{\rho}} [c(v_i) = \rho_i])}. \end{aligned}$$

□

This identity expresses the marginal probability on a permissive block as the function of marginal probabilities on its incident vertices, with modified instances. We now analyze the derivatives of this function.

Lemma 12. *Let $\mathbf{p} = (p_{i,\rho})_{i \in [m], \rho \in \mathcal{F}(B)}$, $\hat{\mathbf{p}} = (\hat{p}_{i,\rho})_{i \in [m], \rho \in \mathcal{F}(B)}$ be two tuples of variables and*

$$f(\mathbf{p}) := \frac{w_{G[B]}(\pi) \prod_{i=1}^m (1 - (1 - \beta)p_{i,\pi})}{\sum_{\rho \in \mathcal{F}(B)} w_{G[B]}(\rho) \prod_{i=1}^m (1 - (1 - \beta)p_{i,\rho})}.$$

Assume for every $i \in [m], \rho \in \mathcal{F}(B(v))$, $p_{i,\rho}, \hat{p}_{i,\rho} \leq \frac{1-\beta}{q-(1-\beta)d_i}$, then

$$|\log f(\mathbf{p}) - \log f(\hat{\mathbf{p}})| \leq \sum_{i \in [d]} \frac{2(1-\beta)}{q - (1-\beta)d_i - 1} \cdot \max_{\rho \in [q]^{B(v)}} |\log p_{i,\rho} - \log \hat{p}_{i,\rho}|.$$

Proof. For every $i \in [m]$, we have

$$\frac{\partial f}{\partial p_{i,\pi}} = -(1-\beta)f(1-f) \cdot \frac{1}{1 - (1-\beta)p_{i,\pi}}.$$

For every $i \in [m]$ and $\rho \neq \pi$, we have

$$\frac{\partial f}{\partial p_{i,\rho}} = (1-\beta)f \cdot \frac{w_{G[B]}(\rho) \prod_{i=1}^m (1 - (1-\beta)p_{i,\rho})}{\sum_{\sigma \in \mathcal{F}(B)} w_{G[B]}(\sigma) \prod_{i=1}^m (1 - (1-\beta)p_{i,\sigma})} \cdot \frac{1}{1 - (1-\beta)p_{i,\rho}}.$$

Thus,

$$\sum_{\substack{\rho \in \mathcal{F}(B) \\ \rho \neq \pi}} \frac{\partial f}{\partial p_{i,\rho}} \leq (1-\beta)f(1-f) \cdot \max_{\substack{\rho \in \mathcal{F}(B) \\ \rho \neq \pi}} \frac{1}{1 - (1-\beta)p_{i,\rho}}.$$

Let $\Phi = \frac{1}{x}$, by mean value theorem, for some $\tilde{\mathbf{p}} = (\tilde{p}_{i,\rho})_{i \in [m], \rho \in [q]^{B(v)}}$ where each $\tilde{p}_{i,\rho} \leq \frac{1-\beta}{q-(1-\beta)d_i}$, we have

$$\begin{aligned} & |\log f(\mathbf{p}) - \log f(\hat{\mathbf{p}})| \\ &= \sum_{i \in [m]} \sum_{\rho \in \mathcal{F}(B)} \left(\frac{\Phi(f)}{\Phi(p_{i,\rho})} \left| \frac{\partial f}{\partial p_{i,\rho}} \right| \right) \Big|_{\mathbf{p}=\tilde{\mathbf{p}}} \cdot |\log p_{i,\rho} - \log \hat{p}_{i,\rho}| \\ &\leq \sum_{i \in [m]} \left(\frac{\Phi(f)}{\Phi(p_{i,\pi})} \left| \frac{\partial f}{\partial p_{i,\pi}} \right| + \sum_{\substack{\rho \in \mathcal{F}(B) \\ \rho \neq \pi}} \frac{\Phi(f)}{\Phi(p_{i,\rho})} \left| \frac{\partial f}{\partial p_{i,\rho}} \right| \right) \Big|_{\mathbf{p}=\tilde{\mathbf{p}}} \cdot \max_{\rho \in [q]^{B(v)}} |\log p_{i,\rho} - \log \hat{p}_{i,\rho}| \\ &\leq \sum_{i \in [m]} \left((1-\beta) \left(\frac{p_{i,\pi}}{1 - (1-\beta)p_{i,\pi}} + \max_{\substack{\rho \in \mathcal{F}(B) \\ \rho \neq \pi}} \frac{p_{i,\rho}}{1 - (1-\beta)p_{i,\rho}} \right) \right) \Big|_{\mathbf{p}=\tilde{\mathbf{p}}} \cdot \max_{\rho \in [q]^{B(v)}} |\log p_{i,\rho} - \log \hat{p}_{i,\rho}| \\ &\leq \sum_{i \in [m]} \frac{2(1-\beta)}{q - (1-\beta)d_i - (1-\beta)} \cdot \max_{\rho \in [q]^{B(v)}} |\log p_{i,\rho} - \log \hat{p}_{i,\rho}|. \end{aligned}$$

□

The following lemma gives an upper bound for the probability $\mathbf{Pr}_{\Omega}[c(v) = x]$.

Lemma 13. Assume $q > (1 - \beta)d$. For every color $x \in [q]$, it holds that

$$\Pr_{\Omega} [c(v) = x] \leq \frac{1}{q - (1 - \beta)d}.$$

where d is the degree of v in G .

Proof. Assume $x = 1$. For every $i \in [q]$, let x_i denote the number of neighbors of v that are of color i . Then $p_{v,1} \leq \max \frac{\beta^{x_1}}{\sum_{i \in [q]} \beta^{x_i}}$ subject to the constraints that all x_i are nonnegative integers and $\sum_{i=1}^q x_i = d$. Since $\beta \leq 1$, we can assume $x_1 = 0$, thus $p_{v,1} \leq \max \frac{1}{1 + \sum_{i=2}^q \beta^{x_i}}$. We now distinguish between two cases:

1. (If $d \geq q - 1$) In this case, let $\lambda = 1 - \beta$, then

$$\frac{1}{1 + \sum_{i=2}^q \beta^{x_i}} \leq \frac{1}{1 + (q-1)(1-\lambda)^{\frac{d}{q-1}}} \stackrel{\heartsuit}{\leq} \frac{1}{1 + (q-1)\left(1 - \frac{\lambda d}{q-1}\right)} = \frac{1}{q - (1 - \beta)d},$$

where \heartsuit is due to the fact that the inequality $(1 - a)^b \geq 1 - ab$ holds when $0 \leq a \leq 1$ and $b \geq 1$.

2. (If $d < q - 1$) In this case, due to the integral constraint of x_i 's, the term $\sum_{i=2}^q \beta^{x_i}$ minimizes when d of x_i 's are set to one and remaining x_i 's are set to zero. Therefore, we have

$$\frac{1}{1 + \sum_{i=2}^q \beta^{x_i}} \leq \frac{1}{1 + d\beta + (q-1-d)} = \frac{1}{q - (1 - \beta)d},$$

□

The recursion (2) holds for arbitrary set of vertices B (not necessary a permissive block), thus if one takes B as a single vertex, it implies the following simple lower bound for marginal probabilities on a vertex.

Lemma 14. For every feasible $x \in [q]$, it holds that

$$\Pr_{\Omega} [c(v) = x] \geq \frac{\beta^d}{q},$$

where d is the degree of v in G .

4 Strong Spatial Mixing

We prove the strong spatial mixing property for Potts model in this section. Recall that

$$\delta(d) = \begin{cases} \frac{2(1-\beta)}{q-1-(1-\beta)d} & \text{if } d \leq \frac{q-1}{1-\beta} - 2 \\ 1 & \text{otherwise.} \end{cases}$$

Theorem 2 is restated in a formal way:

Theorem 15. Let $q \geq 3$ be an integer and $0 \leq \beta < 1$. Let \mathcal{G} be a family of finite graphs that satisfy the followings:

- the function $\delta(\cdot)$ is a contraction function for \mathcal{G} ;

- (proper q -coloring) if $\beta = 0$, then \mathcal{G} is a family of q -colorable graphs.

Then there exist two constants $C_1, C_2 > 0$ such that the following holds: For every graph $G(V, E) \in \mathcal{G}$ with $|V| = n$, every vertex $v \in V$, every color $x \in [q]$, every set of vertices $\Lambda \subseteq V \setminus \{v\}$ and two feasible instances $\Omega_1 = (G, \Lambda, \rho)$, $\Omega_2 = (G, \Lambda, \pi)$ with $\rho, \pi \in [q]^\Lambda$ being two configurations on Λ , it holds that

$$|\mathbf{Pr}_{\Omega_1}[c(v) = x] - \mathbf{Pr}_{\Omega_2}[c(v) = x]| \leq n^{C_1} \cdot \exp(-C_2 \cdot \ell),$$

where $\ell := \text{dist}(v, \Delta)$ and $\Delta \subseteq \Lambda$ is the subset of Λ on which ρ and π differ.

We prove the theorem by using the recursion introduced in Section 3 to estimate marginal probability on a vertex v . In each step, we show that the difference between the (logarithm of) marginal probabilities caused by different configurations on Δ contracts by a factor of $\delta(\cdot)$, and therefore relate the difference of marginal probabilities to the contraction function.

The following observation is useful: If $\delta(\cdot)$ is a contraction function for \mathcal{G} , then for every graph $G \in \mathcal{G}$, every sufficiently long path in G must contain a low-degree vertex. The property is formally stated as:

Lemma 16. *Let \mathcal{G} be a family of finite graphs for which $\delta(\cdot)$ is a contraction function. Then for some constants $\theta > 1$ and $C > 0$, for every $G = (V, E) \in \mathcal{G}$ with $|V| = n$, every $v \in V$ and every $L \geq C \log n$, there exists a low-degree S in $T = T_{\text{SAW}}(G, v)$ such that for every $u \in S$, $L < \text{dist}_T(u, v) \leq \theta L$ and every self-avoiding walk in T from v of length θL intersects S .*

Proof. Let $G(V, E) \in \mathcal{G}$ be a graph. It follows from the definition of contraction function that for some constant $C > 0$, for every $\ell \geq C \log n$, $\mathcal{E}_\delta(v, \ell) < \alpha^\ell$ for some constant $0 < \alpha < 1$.

It is sufficient to show that, for some constant integer $\theta > 0$ it holds that for every $v \in V$, every $L \geq C \log n$, every $P = (v, v_1, \dots, v_{\theta L}) \in \text{SAW}(v, \theta L)$, there exists a low-degree vertex v_j among $\{v_{L+1}, v_{\theta L}, \dots, v_{\theta L}\}$.

Let $\theta = \max \left\{ \lceil \log_{1/\alpha} \left(\frac{q-1}{2(1-\beta)} \right) \rceil, 2 \right\}$. Assume for the contradiction that every vertex in $\{v_{L+1}, v_{L+2}, \dots, v_{\theta L}\}$ has high-degree. Since $\theta L > L \geq C \log n$, we have $\prod_{i=1}^{\theta L} \delta(\deg(v_i)) \leq \alpha^{\theta L}$.

On the other hand, since $\delta(d) \geq \delta(0) = \frac{2(1-\beta)}{q-1}$, we have $\prod_{i=1}^{\theta L} \delta(\deg(v_i)) \geq \left(\frac{2(1-\beta)}{q-1} \right)^L$. This is a contradiction for our choice of θ . \square

4.1 The $\beta > 0$ case

To implement the recursion introduced in Section 3, we define two procedures $\text{marg}(\Omega, v, x, \ell)$ and $\text{marg-block}(\Omega, B(v), \pi, \ell)$ calling each other to estimate vertex and block marginal respectively. We assume $\Omega = (G, \Lambda, \sigma)$ where $G = (V, E)$ is a feasible instance of Potts model, $v \in V \setminus \Lambda$ is a vertex, $x \in [q]$ is a color and ℓ is an integer. Recall that for a permissive block $B(v)$, we use $\mathcal{F}(B)$ to denote the set of feasible configurations over $B(v)$.

Algorithm 1: $\text{marg}(\Omega, v, x, \ell)$

- 1 If v is fixed to be color y , then return 1 if $x = y$ and return 0 if $x \neq y$;
 - 2 If $\ell < 0$ return $1/q$;
 - 3 Compute $B(v)$;
 - 4 For every $\rho \in \mathcal{F}(B(v))$, let $\hat{p}_\rho \leftarrow \text{marg-block}(\Omega, B(v), \rho, \ell)$;
 - 5 Return $\min \left\{ \sum_{\substack{\pi \in \mathcal{F}(B(v)) \\ s.t. \pi(v)=x}} \hat{p}_\pi, \frac{1}{\max\{1, q-(1-\beta)\deg_G(v)\}} \right\}$
-

To describe the algorithm for estimating the block marginals, we need to introduce some notations. Let $B = B(v)$, and we enumerate the boundary edges in δB by $e_i = u_i v_i$ for $i = 1, 2, \dots, m$, where $v_i \notin B$. With this notation more than one u_i or v_i may refer to the same vertex, which is fine. For every $i \in [m]$ and $\rho \in \mathcal{F}(B)$, define Ω_B and Ω_i^ρ as in Lemma 11.

Let $P_i = (v, w_1, w_2, \dots, w_k, v_i)$ be a self-avoiding walk from v to v_i such that all intermediate vertices w_i are in $B(v)$. Since $B(v)$ is a minimal permissive block, such walk always exists, and let P_i be an arbitrary one of them if there are multiple ones.

Algorithm 2: $\text{marg-block}(\Omega, B(v), \pi, \ell)$

- 1 Compute P_i for every $i \in [m]$;
 - 2 $\hat{p}_{i,\rho} \leftarrow \text{marg}(\Omega_i^\rho, v_i, \rho_i, \ell - |P_i|)$ for every $i \in [m]$ and $\rho \in \mathcal{F}(B)$;
 - 3 Return $\frac{w_{G[B]}(\pi) \prod_{i \in [m]} (1 - (1 - \beta) \hat{p}_{i,\pi})}{\sum_{\rho \in \mathcal{F}(B)} w_{G[B]}(\rho) \prod_{i \in [m]} (1 - (1 - \beta) \hat{p}_{i,\rho})}$;
-

We need a few definitions to analyze the two procedures.

Definition 17. Given an instance $\Omega = (G, \Lambda, \sigma)$ of Potts model where $G = (V, E)$, a vertex $v \in V \setminus \Lambda$, a color $x \in [q]$ and an integer ℓ . The computation tree of $\text{marg}(\Omega, v, x, \ell)$, denoted by $\mathcal{CT}(\Omega, v, x, \ell)$, is a rooted tree recursively defined as follows:

- The root of $\mathcal{CT}(\Omega, v, x, \ell)$ is labeled (Ω, v, x, ℓ) ;
- For every recursive call to (Ω', v', x', ℓ') by $\text{marg}(\Omega, v, x, \ell)$ (in the subroutine marg-block), (Ω, v, x, ℓ) has a children which is the computation tree of (Ω', v', x', ℓ') .

Define the *termination set* of $\text{marg}(\Omega, v, x, \ell)$ as the set of vertices u in the self-avoiding walk tree $T_{\text{SAW}}(G[V \setminus \Lambda], v)$ that $\text{marg}(\Omega', u, x', \ell')$ returns at step 2 for some leaf (Ω', u, x', ℓ') of $\mathcal{CT}(\Omega, v, x, \ell)$.

Thus the computation of $\text{marg}(\Omega, v, x, \ell)$ ends either at trivial instance (including vertex with fixed color and one-vertex graph), or at vertices in termination set.

Definition 18. Given an instance $\Omega = (G, \Lambda, \sigma)$ of Potts model with $q \geq 3$ and activity $0 < \beta < 1$ where $G = (V, E)$ with $|V| = n$, a vertex $v \in V \setminus \Lambda$. Let $T = T_{\text{SAW}}(G[V \setminus \Lambda], v)$ be the self-avoiding walk tree rooted at v in $G[V \setminus \Lambda]$ and S be a set of low-degree vertices in T . Assume v has m children v_1, v_2, \dots, v_m in T , let T_i denote the subtree of T rooted at T_i . We recursively define the error function:

$$\mathcal{E}_{T,S} := \begin{cases} \sum_{i=1}^m \delta(\deg_G(v_i)) \cdot \mathcal{E}_{T_i,S} & \text{if } v \notin S, \\ q + n \log \frac{1}{\beta}. & \text{otherwise.} \end{cases}$$

Definition 19. Given an instance $\Omega = (G, \Lambda, \sigma)$ of Potts model where $G = (V, E)$, a vertex $v \in V \setminus \Lambda$, a color $x \in [q]$, an assignment $\pi \in \mathcal{F}(B(v))$ and an integer ℓ . We denote $p_{\Omega,v,\ell}(x) = \text{marg}(\Omega, v, x, \ell)$ and $p_{\Omega,v,B(v),\ell}(\pi) = \text{marg-block}(\Omega, B(v), \pi, \ell)$. Define

$$\begin{aligned} \mathcal{E}_{\Omega,\ell}(v) &:= \max_{x \in [q]} |\log(p_{\Omega,v,\ell}(x)) - \log(\mathbf{Pr}_\Omega[c(v) = x])|; \\ \mathcal{E}_{\Omega,\ell}(B(v)) &:= \max_{\pi \in \mathcal{F}(B(v))} |\log(p_{\Omega,B(v),\ell}(\pi)) - \log(\mathbf{Pr}_\Omega[c(B(v)) = \pi])|. \end{aligned}$$

We use the convention that $\log 0 - \log 0 = 0$.

The following key lemma relates the error functions we introduced above.

Lemma 20. Let $\hat{\Omega} = (\hat{G}, \hat{\Lambda}, \hat{\sigma})$ be an instance of Potts model with $q \geq 3$ and activity $0 < \beta < 1$ where $\hat{G} = (\hat{V}, \hat{E})$. Assume $|\hat{V}| = n$. Let $\hat{v} \in \hat{V} \setminus \hat{\Lambda}$ be a vertex and $x \in [q]$ be a color. Let $L > 0$ be an integer and \hat{S} be the termination set of $\text{marg}(\hat{\Omega}, \hat{v}, x, L)$. Denote $\hat{T} = T_{\text{SAW}}(\hat{G}[\hat{V} \setminus \hat{\Lambda}], \hat{v})$ as the self-avoiding walk tree rooted at \hat{v} in $\hat{G}[\hat{V} \setminus \hat{\Lambda}]$. Then $\mathcal{E}_{\hat{\Omega}, L}(v) \leq \mathcal{E}_{\hat{T}, \hat{S}}$.

Proof. Let $\mathcal{CT} = \mathcal{CT}(\hat{\Omega}, \hat{v}, x, L)$. For every vertex (Ω, v, z, ℓ) in \mathcal{CT} where $\Omega = (G = (V, E), \Lambda, \sigma)$, we apply induction on the depth of $\mathcal{CT}(\Omega, v, z, \ell)$ to show

$$\mathcal{E}_{\Omega, \ell}(v) \leq \mathcal{E}_{T, \hat{S} \cap V(T)},$$

where $T = T_{\text{SAW}}(G[V \setminus \Lambda], v)$ and $V(T)$ is the set of vertices in T .

The base case is that $\text{marg}(\Omega, v, z, \ell)$ is itself a leaf, namely it returns without any further recursive call to marg . Then

- if it is returned at step 1, $\mathcal{E}_{\Omega, \ell}(v) = 0$;
- if it is returned at step 5, $\mathcal{E}_{\Omega, \ell}(v) = 0$;
- if it is returned at step 2, due to Lemma 14, $\mathcal{E}_{\Omega, \ell}(v) \leq q + \deg_G(v) \log \frac{1}{\beta}$.

Assume the lemma holds for smaller depth and $\text{marg}(\Omega, v, z, \ell)$ is not a leaf. In Algorithm 1, the estimation of marginal is computed as:

$$p_{\Omega, v, \ell}(z) = \min \left\{ \sum_{\substack{\pi \in \mathcal{F}(B(v)) \\ \text{s.t. } \pi(v)=z}} p_{\Omega, B(v), \ell}(\pi), \frac{1}{\max\{1, q - (1 - \beta)\deg_G(v)\}} \right\}$$

By Lemma 13, it always holds that $\mathbf{Pr}_{\Omega}[c(v) = z] \leq \frac{1}{\max\{1, q - (1 - \beta)\deg_G(v)\}}$. Thus assuming $p_{\Omega, v, \ell}(z) = \sum_{\substack{\pi \in \mathcal{F}(B(v)) \\ \text{s.t. } \pi(v)=z}} p_{\Omega, B(v), \ell}(\pi)$ will not make the error $\mathcal{E}_{T, S}(v)$ smaller, and hence we have

$$\begin{aligned} \mathcal{E}_{\Omega, \ell}(v) &= \max_{x \in [q]} |\log(\mathbf{Pr}_{\Omega}[c(v) = x]) - \log p_{\Omega, v, \ell}(x)| \\ &= \max_{x \in [q]} \left| \log \left(\sum_{\substack{\pi \in \mathcal{F}(B(v)) \\ \text{s.t. } \pi(v)=x}} \mathbf{Pr}_{\Omega}[c(B(v)) = \pi] \right) - \log \left(\sum_{\substack{\pi \in \mathcal{F}(B(v)) \\ \text{s.t. } \pi(v)=x}} p_{\Omega, B(v), \ell}(\pi) \right) \right| \\ &\leq \max_{\pi \in \mathcal{F}(B(v))} |\log(\mathbf{Pr}_{\Omega}[\sigma(B(v)) = \pi]) - \log(p_{\Omega, B(v), \ell}(\pi))| \\ &= \mathcal{E}_{\Omega, \ell}(B(v)). \end{aligned}$$

where the last inequality is due to that for every positive $a_1, \dots, a_n, b_1, \dots, b_n$, $\frac{\sum_{i \in [n]} a_i}{\sum_{i \in [n]} b_i} \leq \max_{i \in [n]} \frac{a_i}{b_i}$.

Since $\text{marg}(\Omega, v, z, \ell)$ is not a leaf, the value of $p_{\Omega, v, \ell}(z)$ is returned at step 5 of Algorithm 1, and it is computed from the recursion in Algorithm 2. Recall $\delta B(v) = \{u_i v_i \mid i \in [m]\}$. For every $i \in [m]$, we let $\ell_i = \ell - |P_i|$. We claim that Lemma 12 implies

$$\begin{aligned} \mathcal{E}_{\Omega, \ell}(B(v)) &\leq \sum_{i \in [m]} \frac{2(1 - \beta)}{q - (1 - \beta)\deg_G(v_i) - 1} \cdot \max_{\rho \in \mathcal{F}(B(v))} \left| \log(\mathbf{Pr}_{\Omega_i^{\rho}}[\sigma(v_i) = \rho_i]) - \log(p_{\Omega_i^{\rho}, v_i, \ell_i}(\rho_i)) \right| \\ &= \sum_{i \in [m]} \frac{2(1 - \beta)}{q - \deg_G(v_i) - 1} \cdot \max_{\rho \in \mathcal{F}(B(v))} \mathcal{E}_{\Omega_i^{\rho}, \ell_i}(v_i), \end{aligned}$$

where Ω_i^ρ is obtained from Ω as in Lemma 12.

To see this, note all v_i is on the boundary of a permissive block, thus either $\mathcal{E}_{\Omega_i^\rho, \ell - |P_i|}(v_i) = 0$ for every ρ (in case that the color of v_i is fixed), or by Lemma 13,

$$\Pr_{\Omega_i^\rho} [c(v_i) = \rho_i] \leq \frac{1}{q - (1 - \beta)\deg_G(v_i)}.$$

Also from step 5 of Algorithm 1, we have

$$p_{\Omega_i^\rho, v_i, \ell_i}(\rho_i) \leq \frac{1}{q - (1 - \beta)\deg_G(v_i)}.$$

With this upper bound for $\mathcal{E}_{\Omega, \ell}(v)$, and note that every P_i is a self-avoiding walk from v to v_i with every vertex in $B(v)$, we can then apply the induction hypothesis to complete the proof. \square

We are now ready to prove the main theorem of this section.

Proof of Theorem 15 when $\beta > 0$. By the definition of the strong spatial mixing, it is sufficient to prove the theorem for $\ell = \Omega(\log n)$. Let C and θ be the constants in Lemma 16 and assume $\ell > \theta \lceil C \log n \rceil$ be an integer. Let $L = \lfloor \ell / \theta \rfloor \geq C \log n$. Consider $p_{\Omega_1, v, L}(x) = \text{marg}(\Omega_1, v, x, L)$. Let S denote the termination set of $\text{marg}(\Omega_1, v, x, L)$. By our choice of L , Lemma 16 implies that the set S satisfies

1. every vertex in S is of distance $(L, 2L]$ to v in $T = T_{\text{SAW}}(G[V \setminus \Lambda], v)$, i.e., $L < \text{dist}_T(v, u) \leq \theta L$ for every $u \in S$;
2. every path from v to Δ in T intersects S .

It follows from Lemma 20 that

$$|\log(p_{\Omega_1, v, L}(x)) - \log(\Pr_{\Omega_1}[c(v) = x])| \leq \mathcal{E}_{\Omega_1, L}(v) \leq \mathcal{E}_{T, S},$$

and similarly

$$|\log(p_{\Omega_2, v, L}(x)) - \log(\Pr_{\Omega_2}[c(v) = x])| \leq \mathcal{E}_{\Omega_2, L}(v) \leq \mathcal{E}_{T, S}.$$

On the otherhand, we have

$$\mathcal{E}_{T, S} \leq \left(q + \deg_G(v) \log \frac{1}{\beta} \right) \cdot \sum_{k=L+1}^{\theta L} \mathcal{E}_\delta(v, k) \leq n^{C'_1} \cdot \exp(-C'_2 \cdot \ell)$$

for some constants $C'_1, C'_2 > 0$.

Note that by the second property of S , $p_{\Omega_1, v, x, L} = p_{\Omega_2, v, x, L}$, thus

$$\begin{aligned} |\Pr_{\Omega_1}[c(v) = x] - \Pr_{\Omega_2}[c(v) = x]| &\leq |\Pr_{\Omega_1}[c(v) = x] - p_{\Omega_1, v, L}(x)| + |\Pr_{\Omega_2}[c(v) = x] - p_{\Omega_1, v, L}(x)| \\ &\leq n^{C_1} \cdot \exp(-C_2 \cdot \ell) \end{aligned}$$

for some constants $C_1, C_2 > 0$. \square

4.2 The $\beta = 0$ case (Coloring model)

Since the lower bound for marginal probability in Lemma 14 is zero for $\beta = 0$, the quantity $\mathcal{E}_{T,S}$ defined in Definition 18 is no longer bounded above. We slightly modify the procedure **marg** to deal with this case.

Let $\Omega = (G, \Lambda, \sigma)$ be an instance of Potts model with $q \geq 3$ and activity $\beta = 0$ where $G = (V, E)$, $v \in V \setminus \Lambda$ be a vertex, $x \in [q]$ be a color and ℓ be an integer. We define

Algorithm 3: **marg**(Ω, v, x, ℓ)

- 1 If v is fixed to be color y , then return 1 if $x = y$ and return 0 if $x \neq y$;
 - 2 Compute $B(v)$;
 - 3 If $\ell < 0$, then return $1/q$ if there is a feasible $\pi \in \mathcal{F}(B(v))$ such that $\pi(v) = x$ and return 0 if no such π exists;
 - 4 For every $\rho \in \mathcal{F}(B(v))$, let $\hat{p}_\rho \leftarrow \text{marg-block}(\Omega, B(v), \rho, \ell)$;
 - 5 Return $\min \left\{ \sum_{\substack{\pi \in \mathcal{F}(B(v)) \\ s.t. \pi(v)=x}} \hat{p}_\pi, \frac{1}{\max\{1, q - (1-\beta)\deg_G(v)\}} \right\}$
-

The only difference of this version of **marg** is at step 5, where we check whether the color x is locally feasible. We return $1/q$ if so and return 0 otherwise.

Let $T = T_{\text{SAW}}(G[V \setminus \Lambda], v)$ be the self-avoiding walk tree rooted at v in $G[V \setminus \Lambda]$. With our new version of **marg**, define the computation tree $\mathcal{CT} = \mathcal{CT}(\Omega, v, x, \ell)$ the same as in Definition 17, while the termination set of **marg**(Ω, v, x, ℓ) is defined as the set of vertices u in T that **marg**(Ω', u, x', ℓ') returns at step 5 in Algorithm 3 for some leaf (Ω', u, x', ℓ') of \mathcal{CT} .

We can similarly define error functions as the $\beta > 0$ case, with difference on the base case.

Definition 21. Given an instance $\Omega = (G, \Lambda, \sigma)$ of Potts model with $q \geq 3$ and activity $\beta = 0$ where $G = (V, E)$ with $|V| = n$, a vertex $v \in V \setminus \Lambda$. Let $T = T_{\text{SAW}}(G[V \setminus \Lambda], v)$ be the self-avoiding walk tree rooted at v in $G[V \setminus \Lambda]$ and S be a set of low-degree vertices in T . Assume v has m children v_1, v_2, \dots, v_m in T , let T_i denote the subtree of T rooted at T_i . We recursively define the error function:

$$\mathcal{E}_{T,S} := \begin{cases} \sum_{i=1}^m \delta(\deg_G(v_i)) \cdot \mathcal{E}_{T_i,S} & \text{if } v \notin S, \\ n \log q. & \text{otherwise.} \end{cases}$$

Lemma 22. Let $\hat{\Omega} = (\hat{G}, \hat{\Lambda}, \hat{\sigma})$ be an instance of Potts model with $q \geq 3$ and activity $\beta = 0$ where $\hat{G} = (\hat{V}, \hat{E})$. Assume $|\hat{V}| = n$. Let $\hat{v} \in \hat{V} \setminus \hat{\Lambda}$ be a vertex and $x \in [q]$ be a color. Let $L > 0$ be an integer and \hat{S} be the termination set of **marg**($\hat{\Omega}, \hat{v}, x, L$). Denote $\hat{T} = T_{\text{SAW}}(\hat{G}[\hat{V} \setminus \hat{\Lambda}], \hat{v})$ as the self-avoiding walk tree rooted at \hat{v} in $G[\hat{V} \setminus \hat{\Lambda}]$. Then $\mathcal{E}_{\hat{\Omega},L}(v) \leq \mathcal{E}_{\hat{T},\hat{S}}$.

Proof. The proof of this lemma is almost identical to the proof of Lemma 20, except at the base case of the induction.

Consider the situation that (Ω, v, z, ℓ) is a leaf in $\mathcal{CT} = \mathcal{CT}(\hat{\Omega}, \hat{v}, x, L)$, then

- if it is returned at step 1, then $\mathcal{E}_{\Omega,\ell}(v) = 0$;
- if it is returned at step 5, then $\mathcal{E}_{\Omega,\ell}(v) = 0$;
- if it is returned at step 5, then we claim that $\mathcal{E}_{\Omega,\ell}(v) \leq n \log q$. To see this, it is sufficient to show that if there is a feasible $\pi \in \mathcal{F}(B(v))$ such that $\pi(v) = x$, then $\mathbf{Pr}_\Omega[c(v) = x] > 0$, and if no such π exists, then $\mathbf{Pr}_\Omega[c(v) = x] = 0$ (We use the convention that $\log 0 - \log 0 = 0$ and the fact that $\mathbf{Pr}_\Omega[c(v) = x] > 0$ implies $\mathbf{Pr}_\Omega[c(v) = x] \geq q^{-n}$). This is a consequence of Proposition 8

□

Proof of Theorem 15 when $\beta = 0$. With Lemma 20 replaced by Lemma 22, the proof is almost identical to the $\beta > 0$ case. □

5 Approximate Counting and Sampling

In this section, we prove Theorem 3. We first show how to estimate the marginal probability in Potts model and it is routine to obtain FPTAS from this estimation.

5.1 Estimate the marginals

Theorem 23. *Let $q \geq 3$ be an integer and $0 \leq \beta < 1$. Let \mathcal{G} be a family of finite graphs that satisfies the followings:*

- *the function $\delta(\cdot)$ is a contraction function for \mathcal{G} ;*
- *(proper q -coloring) if $\beta = 0$, the family \mathcal{G} is q -colorable;*
- *the family \mathcal{G} is locally sparse.*

Then for every feasible instance $\Omega = (G, \Lambda, \sigma)$ of Potts model where $G = (V, E) \in \mathcal{G}$ with $|V| = n$, $\Lambda \subseteq V$ and $\sigma \in [q]^\Lambda$, for every vertex $v \in V$ and every color $x \in [q]$, there exists an algorithm that can compute an estimation \hat{p} of $\Pr_\Omega[c(v) = x]$ in time polynomial in n , satisfying

$$1 - O\left(\frac{1}{n^3}\right) \leq \frac{\hat{p}}{\Pr_\Omega[c(v) = x]} \leq 1 + O\left(\frac{1}{n^3}\right).$$

Let \mathcal{G} be a family of finite graphs satisfying condition in Theorem 15. Let $\Omega = (G, \Lambda, \sigma)$ be an instance of Potts model where $G = (V, E) \in \mathcal{G}$. Then for every vertex $v \in V$, color $x \in [q]$, set of vertices $\Delta \subseteq V \setminus \{\Lambda \cup \{v\}\}$ and a feasible configuration $\rho \in [q]^\Delta$, we have shown in the proof of Theorem 15 that we can compute an estimate \hat{p} of $\Pr_\Omega[c(v) = x \mid c(\Delta) = \rho]$ such that, for some universal constants $C_1, C_2 > 0$,

$$|\log \hat{p} - \log(\Pr_\Omega[c(v) = x \mid c(\Delta) = \rho])| \leq n^{C_1} \cdot \exp(-C_2 \cdot \ell),$$

where $\ell = \text{dist}_G(v, \Delta)$, as long as $\ell \geq C \log n$ for some constant $C > 0$.

To prove Theorem 23, we show that if \mathcal{G} is locally sparse, then our estimation algorithm is also efficient, i.e., terminates in polynomial time for $L = O(\log |V|)$.

Lemma 24. *Let $q \geq 3$ be an integer and $0 \leq \beta < 1$. Assume \mathcal{G} is a family of graphs satisfying condition in Theorem 23. Then there exists a constant $C > 0$ such that for every feasible instance $\hat{\Omega} = (\hat{G}(\hat{V}, \hat{E}), \hat{\Lambda}, \hat{\sigma})$ with $\hat{G} \in \mathcal{G}$, every vertex $\hat{v} \in \hat{V} \setminus \hat{\Lambda}$, every color $x \in [q]$ and every $L \geq C \log |\hat{V}|$, the procedure $\text{marg}(\hat{\Omega}, \hat{v}, x, L)$ (both $\beta > 0$ and $\beta = 0$ versions) terminates in time $|\hat{V}|^{O(1)} \exp(O(L))$.*

Proof. Let C and θ be the constants in Lemma 16. Fix \hat{S} as the termination set of $\text{marg}(\hat{\Omega}, \hat{v}, x, L)$. Let $\hat{T} = T_{\text{SAW}}(G[\hat{V} \setminus \hat{\Lambda}], v)$. Then it follows from Lemma 16 that $L < \text{dist}_{\hat{T}}(v, \hat{S}) \leq \theta L$. Let $\mathcal{CT} = \mathcal{CT}(\hat{\Omega}, \hat{v}, x, L)$ denote the computation tree of $\text{marg}(\hat{\Omega}, \hat{v}, x, L)$.

For every $(\Omega, v, z, \ell) \in \mathcal{CT}$, where $\Omega = (G(V, E), \Lambda, \sigma)$, consider the self-avoiding walk tree T that is obtained from $T_{\text{SAW}}(G[V \setminus \Lambda], v)$ by removing all descendants of \hat{S} . We use $\mathcal{P}_{\Omega, z}$ to denote the set of self-avoiding walks corresponding to the leaves of T .

We claim that $|\mathcal{P}_{\hat{\Omega}, \hat{v}}| = \exp(O(L))$. To see this, note that $|\mathcal{P}_{\hat{\Omega}, \hat{v}}| \leq \sum_{k=1}^{\theta L} \text{SAW}_{\hat{G}}(\hat{v}, k)$, where $\text{SAW}_{\hat{G}}(\hat{v}, k)$ is the set of self-avoiding walks of length k from \hat{v} in \hat{G} . On the other hand, we have

$$|\hat{V}|^{O(1)} \geq \sum_{k=1}^{\theta L} \mathcal{E}_{\delta}(\hat{v}, k) \geq \left(\frac{2(1-\beta)}{q-1} \right)^{\theta L} \sum_{k=1}^{\theta L} \text{SAW}_{\hat{G}}(\hat{v}, k),$$

where the last inequality is due to $\delta(d) \geq \frac{2(1-\beta)}{q-1}$ for every $d \geq 0$.

The time cost of each vertex $\text{marg}(\Omega, v, z, \ell)$ in \mathcal{CT} besides the recursive calls is at most $C' \cdot mq^{B_{\Omega}(v)}$ for some constant $C' > 0$ where $m = |\delta B_{\Omega}(v)|$ is the size of edge boundary of $B_{\Omega}(v)$. We use $\tau_{\Omega, v}$ to denote the maximum running time of $\text{marg}(\Omega, v, z, \ell)$ over all colors $z \in [q]$. We apply induction on the depth of $\mathcal{CT}(\text{marg}(\Omega, v, z, \ell))$ to show that $\tau_{\Omega, v} \leq C' \sum_{P \in \mathcal{P}_{\Omega, v}} q^{2|\bigcup_{u \in P} B_{\Omega}(u)|}$. If the depth of $\mathcal{CT}(\Omega, v, z, \ell)$ is one, the upper bound is trivial. Now assume the lemma holds for smaller depth. Denote $B = B_{\Omega}(v)$ and assume $\delta B = \{u_i v_i \mid i \in [m]\}$ be the edge boundary of B . Then

$$\tau_{\Omega, v} \leq \sum_{\pi \in \mathcal{F}(B)} \sum_{i \in [m]} \tau_{\Omega_i^{\pi}, v_i} + C' \cdot mq^{|B|} \leq q^{|B|} \sum_{i \in [m]} \left(C' + \max_{\pi \in \mathcal{F}(B)} \tau_{\Omega_i^{\pi}, v_i} \right)$$

Applying the induction hypothesis, we have for some $\pi \in \mathcal{F}(B)$

$$\begin{aligned} \tau_{\Omega, v} &\leq C' \cdot q^{|B|} \cdot \sum_{i \in [m]} \left(1 + \sum_{P \in \mathcal{P}_{\Omega_i^{\pi}, v_i}} q^{2|\bigcup_{u \in P} B_{\Omega_i^{\pi}}(u)|} \right) \\ &\leq C' \sum_{i \in [m]} \sum_{P \in \mathcal{P}_{\Omega_i^{\pi}, v_i}} q^{2(|\bigcup_{u \in P} B_{\Omega_i^{\pi}}(u)| + |B|)} \\ &\leq C' \cdot \sum_{P \in \mathcal{P}_{\Omega, v}} q^{2|\bigcup_{u \in P} B_{\Omega}(u)|}, \end{aligned}$$

where the last inequality is due to the following three facts:

1. each path in $\mathcal{P}_{\Omega_i^{\pi}, v_i}$ is a part of some path in $\mathcal{P}_{\Omega, v}$, and all these paths in $\mathcal{P}_{\Omega, v}$ are distinct;
2. $B \cap B_{\Omega_i^{\pi}}(u) = \emptyset$ for every i, u and π ;
3. $B_{\Omega}(u)$ is at least as large as $B_{\Omega_i^{\pi}}(u)$ for every u .

Therefore, we have

$$\tau_{\hat{\Omega}, \hat{v}} \leq C' \cdot \sum_{P \in \mathcal{P}_{\Omega, v}} q^{2|\bigcup_{u \in P} B_{\Omega}(u)|} = |\hat{V}|^{O(1)} \exp(O(L))$$

□

Proof of Theorem 23. Assume $|V| = n$. Let C and θ be the constants in Lemma 16. Fix some $L \geq C \log n$ and denote S the termination set of $\text{marg}(\Omega, v, x, L)$. Let $p_{\Omega, v, L}(x) = \text{marg}(\Omega, v, x, L)$ and $T = T_{\text{SAW}}(G[V \setminus \Lambda], v)$. Then it follows from Lemma 20 and Lemma 22 that

$$|\log(p_{\Omega, v, L}(x)) - \log(\mathbf{Pr}_{\Omega}[c(v) = x])| \leq \mathcal{E}_{T, S}.$$

We also have

$$\mathcal{E}_{T,S} \leq n^{O(1)} \cdot \sum_{k=L+1}^{\theta L} \mathcal{E}_\delta(v, k) \leq n^{C_1} \cdot \exp(-C_2 \cdot L)$$

for some universal constants $C_1, C_2 > 0$.

Thus for some $L = O(\log n)$ and $L \geq C \log n$, it holds that

$$1 - O\left(\frac{1}{n^3}\right) \leq \frac{\hat{p}_{\Omega, v, S_v}(x)}{\mathbf{Pr}_\Omega[c(v) = x]} \leq 1 + O\left(\frac{1}{n^3}\right).$$

The running time of the algorithm directly follows from Lemma 24. \square

5.2 The sampling algorithm

Theorem 25. *Let \mathcal{G} be a family of graphs satisfying the conditions in Theorem 23. There exists an FPTAS to compute the partition function of Potts model with parameter q and β for every graph in \mathcal{G} .*

Proof. Let $\Omega = (G, \emptyset, \emptyset)$ be an instance of Potts model, where $G(V, E) \in \mathcal{G}$. Without loss of generality, we give an algorithm to compute an approximation of the partition function $\hat{Z}(\Omega)$ satisfying

$$1 - O\left(\frac{1}{n^2}\right) \leq \frac{\hat{Z}(\Omega)}{Z(\Omega)} \leq 1 + O\left(\frac{1}{n^2}\right).$$

Since our family of instances of Potts model is “self-embeddable” in the sense of [SJ89], the algorithm can be boosted into an FPTAS.

Assume $V = \{v_1, \dots, v_n\}$. First find a configuration $\sigma \in [q]^V$ such that $w_G(\sigma) > 0$. This task is trivial when $\beta > 0$. When $\beta = 0$, since G is q -colorable, we can also do it in polynomial time:

- If the graph is not empty, then choose a vertex v and find a feasible coloring of $B(v)$. Then remove $B(v)$ from the graph and repeat the process.

If G is q -colorable, then $G[V \setminus B(v)]$ is colorable as the boundary of $B(v)$ consists of low-degree vertices, thus the above process will end with a proper coloring of G , which is the union of colorings found at each step. The process terminates in polynomial time since \mathcal{G} is locally sparse and thus the size of every $B(v)$ is $O(\log n)$.

With σ in hand, we have

$$\begin{aligned} Z(\Omega) &= w_G(\sigma) / \mathbf{Pr}_\Omega[c(V) = \sigma] = w_G(\sigma) \left(\mathbf{Pr}_\Omega \left[\bigwedge_{i=1}^n c(v_i) = \sigma(v_i) \right] \right)^{-1} \\ &= w_G(\sigma) \left(\prod_{i=1}^n \mathbf{Pr}_\Omega \left[c(v_i) = \sigma(v_i) \mid \bigwedge_{j=1}^{i-1} c(v_j) = \sigma(v_j) \right] \right)^{-1} \end{aligned}$$

For every $i \in [n]$, let $\Omega_i = (G, \Lambda_i, \sigma_i)$ where $\Lambda_i = \{v_1, \dots, v_{i-1}\}$ and $\sigma_i(v_j) = \sigma(v_j)$ for every $j = 1, \dots, i-1$. We have

$$Z(\Omega) = w_G(\sigma) \left(\prod_{i=1}^n \mathbf{Pr}_{\Omega_i}[c(v_i) = \sigma(v_i)] \right)^{-1}.$$

Note that the graph class \mathcal{G} is closed under the operation of fixing some vertex to a specific color, we can apply Theorem 23 for every Ω_i and obtain \hat{p}_i such that

$$1 - O\left(\frac{1}{n^3}\right) \leq \frac{\hat{p}_i}{\Pr_{\Omega_i}[c(v_i) = \sigma(v_i)]} \leq 1 + O\left(\frac{1}{n^3}\right).$$

Let $\hat{Z}(\Omega) = w_G(\sigma) (\prod_{i=1}^n \hat{p}_i)^{-1}$, then Theorem 23 implies that

$$1 - O\left(\frac{1}{n^2}\right) \leq \frac{\hat{Z}(\Omega)}{Z(\Omega)} \leq 1 + O\left(\frac{1}{n^2}\right).$$

□

Our approximate counting algorithm implies a sampling algorithm via Jerrum-Valiant-Vazirani reduction[JVV86].

Corollary 26. *Let $q > 2$ and $0 \leq \beta < 1$ be two constants. For a family of graphs \mathcal{G} satisfying conditions in Theorem 23, and every graph $G(V, E) \in \mathcal{G}$ with $|V| = n$, there exists an algorithm \mathcal{S} such that for any $\epsilon > 0$ with high probability \mathcal{S} returns a random configuration in $[q]^{V(G)}$ from a distribution that is within total variation distance ϵ from the Gibbs distribution μ_G for the q -state Potts model with activity β . And the running time of \mathcal{S} is in polynomial in n and $\log \frac{1}{\epsilon}$. When $\beta = 0$, i.e. for $q > 3d + 4$, with high probability \mathcal{S} is an FPAUS (fully polynomial-time almost uniform sampler) for proper q -colorings of G .*

6 Random Graphs

In this section, we prove Theorem 4. We first prove the following properties of $\mathcal{G}(n, d/n)$.

Theorem 27. *Let d be a sufficiently large constant, $q > 3(1 - \beta) + 4$ and $G = (V, E) \sim \mathcal{G}(n, d/n)$. Then with probability $1 - o(1)$, the following holds*

- *there exist two universal positive constants $C > 0, \gamma < 1$ such that $\mathcal{E}_\delta(v, \ell) < n^C \gamma^\ell$ for all $v \in V$ and for all $\ell = o(\sqrt{n})$;*
- *if $\beta = 0$, then G is q -colorable;*
- *there exists a universal constant $C > 0$ such that for every path P in G of length ℓ , $|B(P)| \leq C(\ell + \log n)$.*

Note that the first property in above theorem impose an upper bound on ℓ . This is not harmful as our algorithms for FPTAS and sampling only require the property holds for $\ell = O(\log n)$. Thus Theorem 27 and Corollary 26 together imply Theorem 4.

It is well-known that when $\beta = 0$, G is q -colorable with high probability (see e.g., [GM75]), we verify the first property in Lemma 28 and the third property in Lemma 30.

6.1 Correlation decay in random graphs

Lemma 28. *Let $d > 1$, $0 \leq \beta < 1$ and $q > 3(1 - \beta)d + 4$ be constants. Let $G(V, E) \sim \mathcal{G}(n, d/n)$. There exist two positive constants $C > 0$ and $\gamma < 1$ such that with probability $1 - O\left(\frac{1}{n}\right)$, for every $v \in V$ and every $\ell = o(\sqrt{n})$, it holds that*

$$\mathcal{E}_\delta(v, \ell) \leq n^C \gamma^\ell$$

We first prove a technical lemma.

Lemma 29. *Let $0 \leq \beta < 1$ be a constant. Let $f_q(d) : \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}^{\geq 0}$ be a piece wise function defined as*

$$f_q(d) := \begin{cases} \frac{2(1-\beta)}{q-1-(1-\beta)d} & \text{if } d \leq \frac{q-1}{1-\beta} - 2 \\ 1 & \text{otherwise.} \end{cases}$$

Let X be a random variable distributed according to binomial distribution $\text{Bin}(n, \frac{\Delta}{n})$ where $\Delta > 1$ is a constant. Then for $q \geq 3(1-\beta)\Delta + 2$ and all sufficiently large n , it holds that $\mathbf{E}[f_q(X)] < \frac{1}{\Delta}$.

Proof. Let $\lambda = 1 - \beta$. Since $f(d)$ is decreasing in q , we can assume $q = 3\lambda\Delta + 2$. Note that

$$\mathbf{E}_{d \sim \text{Bin}(n, \frac{\Delta}{n})} [f(d)] \leq \frac{1}{\Delta} \iff \mathbf{E}_{d \sim \text{Bin}(n, \frac{\Delta}{n})} [1 - f(d)] \geq \frac{\Delta - 1}{\Delta}.$$

Let $g(x) := 1 - f(x)$, then

$$\mathbf{E}_{d \sim \text{Bin}(n, \frac{\Delta}{n})} [1 - f(d)] = \sum_{k=0}^{\lfloor \frac{q-1}{\lambda} - 2 \rfloor} g(k) \cdot p(k)$$

where $p(k) = \binom{n}{k} \left(\frac{\Delta}{n}\right)^k \left(1 - \frac{\Delta}{n}\right)^{n-k}$.

Define

$$\begin{aligned} \tilde{g}(x) := & 1 - \frac{2\lambda}{q-1-\lambda\Delta} - \frac{2\lambda^2(x-\Delta)}{(q-1-\lambda\Delta)^2} - \frac{2\lambda^3(x-\Delta)^2}{(q-1-\lambda\Delta)^3} - \frac{2\lambda^4(x-\Delta)^3}{(q-1-\lambda\Delta)^4} \\ & - \frac{2\lambda^5(x-\Delta)^4}{(q-1-\lambda\Delta)^5} - \frac{2\lambda^6(x-\Delta)^5}{(q-1-\lambda\Delta)^6} - \frac{2\lambda^6(x-\Delta)^6}{(q-1-\lambda\Delta)^6}. \end{aligned}$$

Then

$$g(x) - \tilde{g}(x) = \frac{2\lambda^6(q-1-\lambda-x\lambda)(x-\Delta)^6}{(q-1-x\lambda)(q-1-\lambda\Delta)^6},$$

which is positive for $x \leq \lfloor \frac{q-1}{\lambda} - 2 \rfloor$.

We now prove that

$$\sum_{k=0}^{\lfloor \frac{q-1}{\lambda} - 2 \rfloor} \tilde{g}(k) \cdot p(k) \geq \frac{\Delta - 1}{\Delta}.$$

The expectation of $\tilde{g}(k)$ can be computed directly:

$$\mathbf{E}[\tilde{g}(k)] = \frac{1}{n^5(q-1-\lambda\Delta)^6} \cdot (C_5 n^5 + C_4 n^4 \pm O(n^3)),$$

where

$$\begin{aligned} C_5 = & 1 - 2\lambda + (12\lambda - 20\lambda^2 - 2\lambda^3 - 2\lambda^4 - 2\lambda^5 - 4\lambda^6)\Delta \\ & + (60\lambda^2 - 80\lambda^3 - 12\lambda^4 - 14\lambda^5 - 74\lambda^6)\Delta^2 + (160\lambda^3 - 160\lambda^4 - 24\lambda^5 - 50\lambda^6)\Delta^3 \\ & + (240\lambda^4 - 160\lambda^5 - 16\lambda^6)\Delta^4 + (192\lambda^5 - 64\lambda^6)\Delta^5 + 64\lambda^6\Delta^6; \\ C_4 = & 2\lambda^3(1 + 3\lambda + 7\lambda^2 + 46\lambda^3)\Delta^2 + 2\lambda^3(6\lambda + 18\lambda^2 + 234\lambda^3)\Delta^3 \\ & + 2\lambda^3(12\lambda^2 + 69\lambda^3)\Delta^4 + 16\lambda^6\Delta^5. \end{aligned}$$

Since $C_4 > 0$, thus for sufficiently large n , it holds that

$$\mathbf{E} [\tilde{g}(x)] \geq \frac{C_5}{(q-1-\lambda\Delta)^6}.$$

We also have that

$$\mathbf{E} [\tilde{g}(x)] = \sum_{k=0}^{\lfloor \frac{q-1}{\lambda} - 2 \rfloor} \tilde{g}(k) \cdot p(k) + \sum_{k=\lfloor \frac{q-1}{\lambda} - 1 \rfloor}^n \tilde{g}(k) \cdot p(k)$$

It can be verified that $\tilde{g}(x)$ is monotonically decreasing in x when $x \geq \frac{q-1}{\lambda} - 2$ and $\tilde{g}\left(\frac{q-1}{\lambda} - 2\right) = -\left(\frac{1+2\lambda(\Delta-1)}{1+2\lambda\Delta}\right)^6 < 0$.
Thus we have

$$\sum_{k=0}^{\lfloor \frac{q-1}{\lambda} - 2 \rfloor} \tilde{g}(k) \cdot p(k) \geq \mathbf{E} [\tilde{g}(x)] \geq \frac{C_5}{(q-1-\lambda\Delta)^6} = \frac{\Delta-1}{\Delta} + h(\Delta)$$

where

$$\begin{aligned} h(\Delta) = & (1 + 10\lambda\Delta + (40\lambda^2 - 2\lambda^3 - 2\lambda^4 - 2\lambda^5 - 4\lambda^6)\Delta^2 \\ & + (80\lambda^3 - 12\lambda^4 - 14\lambda^5 - 74\lambda^6)\Delta^3 + (80\lambda^4 - 24\lambda^5 - 50\lambda^6)\Delta^4) \\ & + (32\lambda^5 - 16\lambda^6)\Delta^5) \cdot (\Delta(1 + 2\lambda\Delta)^6)^{-1}. \end{aligned}$$

It can be verified that $h(\Delta)$ is positive for every $0 < \lambda < 1$ and $\Delta \geq 1$. \square

Proof of Lemma 28. Let $v \in V$ be arbitrary fixed and $T_v = T_{\text{SAW}}(G, v)$ and $\ell > 0$ be an integer. By linearity of expectation, we have

$$\mathbf{E} [\mathcal{E}_\delta(v, \ell)] \leq n^\ell \left(\frac{d}{n}\right)^\ell \mathbf{E} \left[\prod_{i=1}^{\ell} \delta(\deg_G(v_i)) \mid P = (v, v_1, \dots, v_\ell) \text{ is a path} \right].$$

Fix a tuple $P = (v, v_1, \dots, v_\ell)$. To calculate the expectation, we construct an independent sequence whose product dominates $\prod_{i=1}^{\ell} \delta(\deg_G(v_i))$ as follows.

Conditioning on $P = (v, v_1, \dots, v_\ell)$ being a path in G . Let X_1, X_2, \dots, X_ℓ be random variables such that each X_i represents the number of edges between v_i and vertices in $V \setminus \{v_1, \dots, v_\ell\}$; and let Y be a random variable representing the number of edges between vertices in $\{v_1, \dots, v_\ell\}$ except for the edges in the path $P = (v, v_1, \dots, v_\ell)$. Then X_1, \dots, X_ℓ, Y are mutually independent binomial random variables with each X_i distributed according to $\text{Bin}(n - \ell, \frac{d}{n})$ and Y distributed according to $\text{Bin}(\binom{\ell}{2} - \ell + 1, \frac{d}{n})$, and for each v_i in the path we have $\deg_G(v_i) = X_i + 2 + Y_i$ with some $Y_1 + Y_2 + \dots + Y_\ell = 2Y$.

Note that $\delta(\deg_G(v_i)) = f_q(\deg_G(v_i))$ where the function $f_q(x)$ is defined in Lemma 29. Note that the ratio $f_q(x)/f_q(x-1)$ is always upper bounded by 2, and we have $f_q(x+1) \leq f_{q-1}(x)$. Thus, conditioning on that $P = (v, v_1, \dots, v_\ell)$ is a path, the product $\prod_{i=1}^{\ell} \delta_{q,\beta}(\deg_G(v_i))$ can be bounded as follows:

$$\prod_{i=1}^{\ell} \delta(\deg_G(v_i)) = \prod_{i=1}^{\ell} f_q(X_i + Y_i + 2) \leq 2^{2Y} \prod_{i=1}^{\ell} f_{q-2}(X_i).$$

Let $d' = \frac{q-4}{3(1-\beta)}$, then we have $d' > d$. Let X be a binomial random variable distributed according to $\text{Bin}(n, \frac{d'}{n})$, thus X probabilistically dominates every X_i whose distribution is $\text{Bin}(n - \ell, \frac{d}{n})$. Since $X_1, X_2, \dots, X_\ell, Y$ are mutually independent conditioning on $P = (v, v_1, \dots, v_\ell)$ being a path in G , for any $P = (v, v_1, \dots, v_\ell)$ we have

$$\mathbf{E} \left[\prod_{i=1}^{\ell} \delta(\deg_G(v_i)) \mid P \text{ is a path} \right] \leq \mathbf{E} \left[4^Y \prod_{i=1}^{\ell} f_{q-2}(X_i) \right] \leq \mathbf{E} [4^Y] \mathbf{E} [f_{q-2}(X)]^\ell.$$

Recall that $Y \sim \text{Bin} \left(\binom{\ell}{2} - \ell + 1, \frac{d}{n} \right)$, the expectation $\mathbf{E} [4^Y]$ can be bounded as

$$\mathbf{E} [4^Y] \leq \sum_{k=0}^{\ell^2} 4^k \binom{\ell^2}{k} \left(\frac{d}{n} \right)^k \left(1 - \frac{d}{n} \right)^{\ell^2 - k} = \left(1 + \frac{3d}{n} \right)^{\ell^2} \leq \exp \left(\frac{3d\ell^2}{n} \right).$$

Since $q - 2 \geq 3(1 - \beta)d' + 2$, it follows from Lemma 29 that $\mathbf{E} [f_{q-2}(X)] \leq \frac{1}{d'} = \frac{3(1-\beta)}{q-4}$. Therefore,

$$\mathbf{E} \left[\prod_{i=1}^{\ell} \delta(\deg_G(v_i)) \mid P \text{ is a path} \right] \leq \exp \left(\frac{3d\ell^2}{n} \right) \left(\frac{3(1-\beta)}{q-4} \right)^\ell \leq \frac{1}{d^\ell} \cdot \exp \left(-\ell \log \left(\frac{q-4}{3d(1-\beta)} \right) + \frac{3d\ell^2}{n} \right).$$

Since $\ell = o(\sqrt{n})$,

$$\mathbf{E} [\mathcal{E}_\delta(v, \ell)] \leq \exp \left(-\ell \log \left(\frac{q-4}{3d(1-\beta)} \right) + o(1) \right).$$

Then the lemma follows from the Markov inequality and the union bound. \square

6.2 Locally sparse for random graphs

Lemma 30. *Let $\varepsilon > 0$ be some fixed constant. Let d be a sufficiently large number, $q \geq (2 + \varepsilon)d$ and $0 \leq \beta < 1$ be constants. Let $G = (V, E) \sim \mathcal{G}(n, d/n)$. There exists a constant $C > 0$ such that with probability $1 - O(\frac{1}{n})$, for every path P in G of length ℓ , $|B(P)| \leq C(\ell + \log n)$.*

Given $P = (v_1, \dots, v_L)$, we are going to upper bound the probability

$$\mathbf{Pr} [|B(P)| \geq t \mid P \text{ is a path}] \tag{3}$$

for every $t > 0$.

A vertex v is a high-degree vertex if $\deg_G(v) \geq \frac{q-1}{1-\beta} - 2$. Thus the probability (3) is maximized when $\beta = 0$. Note that conditioning on P is a path gives each vertex at most two degrees, we can redefine the notion of “high-degree” as $\deg_G(v) \geq q - 5$ and drop the condition that P is a path. Thus it is sufficient to upper bound

$$\mathbf{Pr} [|B(P)| \geq t]$$

with our new definition of high-degree vertices.

Let $G = (V, E)$ be a graph. We now describe a BFS procedure to generate $B^*(P) := B(P) \cup \partial B(P)$. Since $B^*(P)$ is always a superset of $B(P)$, it is sufficient to bound $\mathbf{Pr} [|B^*(P)| \geq t]$. For a vertex $v \in V$, we use $N_G(v)$ to denote the set of neighbors of v in G .

Initially, we have a counter $i = 0$, a graph $G_0 = G$, a set of active vertices $\mathcal{A}_0 = \{v_1, v_2, \dots, v_L\}$ and a set of used vertices $\mathcal{U}_0 = \emptyset$.

(P1)

1. Increase the counter i by one.
2. (If $i \leq L$) Define $G_i(V_i, E_i) = G_{i-1}[V_{i-1} \setminus \{v_i\}]$. Let $\mathcal{U}_i = \mathcal{U}_{i-1} \cup \{v_i\}$. Let $\mathcal{A}_i = (\mathcal{A}_{i-1} \cup N_{G_{i-1}}(v_i)) \setminus \mathcal{U}_i$. Goto 1.
3. (If $i > L$) Terminate if $\mathcal{A}_{i-1} = \emptyset$. Otherwise, let $u \in \mathcal{A}_{i-1}$ and let $\mathcal{U}_i = \mathcal{U}_{i-1} \cup \{u\}$.
 - (a) (If $|N_G(u)| \geq q-5$) Define $G_i(V_i, E_i) = G_{i-1}[V_{i-1} \setminus \{u\}]$. Let $\mathcal{A}_i = (\mathcal{A}_{i-1} \cup N_{G_{i-1}}(u)) \setminus \mathcal{U}_i$. Goto 1.
 - (b) (If $|N_G(u)| < q-5$) Define $G_i = G_{i-1}$. Let $\mathcal{A}_i = \mathcal{A}_{i-1} \setminus \mathcal{U}_i$. Goto 1

The following proposition is immediate:

Proposition 31. *Assume the algorithm terminates at step t , then $B^*(P) = \mathcal{U}_{t-1}$ and $|B^*(P)| = t-1$*

Let $R = \{r_1, r_2, \dots, r_L\}$ be a set and each r_i is the root of tree T_i . We now describe a BFS procedure to explore these L trees. For a vertex v , we use $C(v)$ to denote its children.

Initially, we have a counter $i = 0$ and a set of active vertices $\mathcal{B}_0 = R$.

(P2)

1. Increase the counter i by one.
2. (If $i \leq L$) Let $\mathcal{B}_i = (\mathcal{B}_{i-1} \cup C(r_i)) \setminus \{r_i\}$. Goto 1.
3. (If $i > L$) Terminate if $\mathcal{B}_{i-1} = \emptyset$. Otherwise, let $w \in \mathcal{B}_{i-1}$
 - (a) (If $|C(w)| \geq \frac{q-5}{2}$) Let $\mathcal{B}_i = (\mathcal{B}_{i-1} \cup C(w)) \setminus \{w\}$. Goto 1.
 - (b) (If $|C(w)| < \frac{q-5}{2}$) Let $\mathcal{B}_i = \mathcal{B}_{i-1} \setminus \{w\}$. Goto 1.

Now assume $G \sim \mathcal{G}(n, d/n)$ and for every $i \in [L]$, T_i is a branching process with distribution $\text{Bin}(n, d/n)$, i.e., each $C(u) \sim \text{Bin}([n], d/n)$. We can implement the (P1) when at each step i , the vertex u chosen from the active set sample its neighbors $N_{G_{i-1}}(u)$ according to $\text{Bin}(V_i, d/n)$. This random process can be coupled with $\mathcal{G}(n, d/n)$ such that $B^*(P)$ found by it is always a superset of the one in $\mathcal{G}(n, d/n)$.

We now construct a coupling of (P1) and (P2) with the property that the later one always terminates no earlier than the former one.

At each step $i \geq 1$, let u and w be the vertex chosen from \mathcal{A}_i and \mathcal{B}_i respectively ($u = v_i$ and $w = r_i$ if $i \leq L$). Then $|N_{G_{i-1}}(u)| \sim \text{Bin}(|V_i|, d/n)$. We couple it with some $x \sim \text{Bin}(n, d/n)$ with the property that $x \geq |N_{G_{i-1}}(v_i)|$ and let $C(w)$ be a set with x elements.

Lemma 32. *For every $i \geq 0$, the following two properties hold:*

- (i1) *There exists a surjective mapping F_i from \mathcal{B}_i to \mathcal{A}_i in each step i .*
- (i2) *For every $u \in \mathcal{A}_i$, we use $n_i(u)$ to denote the number of $w \in \mathcal{B}_i$ such that $F_i(w) = u$. Then for every $u \in \mathcal{A}_i$, $n_i(u) \geq |N_G(u)| - |N_{G_i}(u)|$.*

Proof. We apply induction on i to prove the lemma.

When $i = 0$, we let $F_0 : \mathcal{B}_0 \rightarrow \mathcal{A}_0$ be the function that $F_0(r_j) = v_j$ for every $j \in [L]$. Then both properties hold trivially.

Assume the lemma holds for smaller i . If $i \leq L$, since by our coupling, $|C(r_i)| \geq |N_{G_{i-1}}(v_i)|$, we can construct F_i by extending F_{i-1} with an arbitrary surjective mapping from $C(r_i)$ to $N_{G_{i-1}}(v_i)$. For every $u' \in \mathcal{A}_i$, if $u' \in N_{G_{i-1}}(v_i)$, then $n_i(u') \geq n_{i-1}(u') + 1$ and $|N_{G_{i-1}}(u)| - |N_{G_i}(u')| = 1$; otherwise $n_i(u') = n_{i-1}(u')$ and $|N_{G_{i-1}}(u)| = |N_{G_i}(u')|$. Induction hypothesis implies both (i1) and (i2) hold.

If $i > L$, we have to distinguish between cases:

- (If $|N_G(u)| \geq q - 5$ and $N_{G_{i-1}}(u) \geq \frac{q-5}{2}$) We construct F_i by extending F_{i-1} with an arbitrary surjective mapping from $C(w)$ to $N_{G_{i-1}}(u)$, the same argument as $i \leq L$ case proves (i1) and (i2).
- (If $|N_G(u)| \geq q - 5$ and $N_{G_{i-1}}(u) < \frac{q-5}{2}$) In this case, by induction hypothesis, we know that

$$n_{i-1}(u) \geq |N_G(u)| - |N_{G_{i-1}}(u)| \geq \frac{q-5}{2} > N_{G_{i-1}}(u).$$

Choose a surjective f from $F_{i-1}^{-1}(u)$ to $N_{G_{i-1}}(u)$ and construct F_i from F_{i-1} by replacing the mapping on $F_{i-1}^{-1}(u)$ by f . This is safe since $u \notin \mathcal{A}_i$. The same argument as before proves (i1) and (i2).

- (If $|N_G(u)| < q - 5$) Construct $F_i = F_{i-1}$. Since everything does not change, the induction hypothesis implies (i1) and (i2).

□

The first property above guarantees that (P2) terminates no earlier than (P1) and thus its stopping time is an upper bound for the size of $B^*(P)$ found by (P1).

(P2) can be modeled as follows:

1. Let $X \sim \text{Bin}(n, d/n)$ and X_1, X_2, \dots be an infinite sequence of independent random variables defined as follows
 - For $i = 1, 2, \dots, L$, X_i is an independent copy of X ;
 - For $i > L$, X_i has following distribution

$$X_i = \begin{cases} 0 & \text{if } X < (q-5)/2 \\ X & \text{otherwise.} \end{cases}$$

2. Y_1, Y_2, \dots is an infinite sequence of random variables that $Y_0 = L$ and $Y_i = Y_{i-1} + X_i - 1$ for every $i \geq 1$.
3. $Z = \min_t \{Y_t = 0\}$.

The above process is identical to (P2), thus we have

Proposition 33. (P2) terminates after step t if and only if $Z > t$.

Note that $Z > t$ implies $Y_t \geq 0$, we turn to bound the latter.

Lemma 34. *There exist two constants $C_1, C_2 > 0$ depending on d and ε such that*

$$\Pr[Y_t \geq 0] \leq \exp(-C_1 t + C_2 L).$$

Proof. By the definition, $Y_{t+L} = L - (t + L) + \sum_{i=1}^{L+t} X_i = -t + \sum_{i=1}^L X_i + \sum_{i=L+1}^{L+t} X_i$. We know the distribution of X_i s and we now compute their moment generating function. For every $s > 0$, it holds that

$$\Pr[Y_{t+L} \geq 0] = \Pr[e^{sY_{t+L}} \geq 1] \leq \mathbf{E}[e^{sY_{t+L}}] = e^{-st} (\mathbf{E}[e^{sX}])^L (\mathbf{E}[e^{sX_{L+1}}])^t.$$

Recall that $X \sim \text{Bin}(n, d/n)$, we have $\mathbf{E}[e^{sX}] = (1 + \frac{d}{n}(e^s - 1))^n \leq e^{d(e^s - 1)}$. Let $p = (q - 5)/2$, we have

$$\begin{aligned} \mathbf{E}[e^{sX_{L+1}}] &= \Pr[X < p] + \sum_{k=\lfloor p \rfloor}^n e^{sk} \cdot \Pr[X = k] \\ &\leq 1 + \sum_{k=\lfloor p \rfloor}^n e^{sk} \cdot \Pr[X \geq k] \\ &\leq \exp\left(\sum_{k=\lfloor p \rfloor}^{\infty} e^{sk} \cdot \Pr[X \geq k]\right) \end{aligned}$$

By Chernoff bound, for sufficiently large d , we have for some choices of $s > 0$ and $C_1 > 0$,

$$\sum_{k=\lfloor p \rfloor}^{\infty} e^{sk} \cdot \Pr[X \geq k] - s < -C'_1.$$

Let $C'_2 = d(e^s - 1)$, we have

$$\Pr[Y_{t+L} \geq 0] \leq \exp(-C'_1 t + C'_2 L).$$

This implies for some constants $C_1, C_2 > 0$,

$$\Pr[Y_t \geq 0] \leq \exp(-C_1 t + C_2 L).$$

□

Proof of Lemma 30. By Lemma 34 and the union bound, the probability that there exists a path P in G of length ℓ such that $|B(P)| \geq t$ is upper bounded by

$$n \cdot n^\ell \cdot \left(\frac{d}{n}\right)^\ell \cdot \Pr[Y_t \geq 0] \leq n \cdot d^\ell \cdot \exp(-C_1 t + C_2 \ell) = O\left(\frac{1}{n}\right)$$

for $t = C(\ell + \log n)$ and sufficiently large constant C .

□

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